As another example, use (14.13b) to calculate $\boldsymbol{d}(\boldsymbol{u\sigma})$, where $\boldsymbol{u}$ is a vector-valued 0-form (vector) and $\boldsymbol{\sigma}$ is a scalar-valued 1-form (1-form):

$$\boldsymbol{d}(\boldsymbol{u\sigma}) = (\boldsymbol{du}) \wedge \boldsymbol{\sigma} + \boldsymbol{u}\, \boldsymbol{d\sigma}.$$

If one were following the practice of earlier chapters, one would have written $\boldsymbol{u} \otimes \boldsymbol{\sigma}$ where $\boldsymbol{u\sigma}$ appears here, $\boldsymbol{u} \otimes \boldsymbol{d\sigma}$ instead of $\boldsymbol{u}\, \boldsymbol{d\sigma}$, and $\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta$ instead of $\boldsymbol{e}_\alpha \boldsymbol{e}_\beta$. However, to avoid overcomplication in the notation, all such tensor product symbols are omitted here and hereafter.

Equations (14.12) and (14.13) do more than define the (extended) exterior derivative $\boldsymbol{d}$ and provide a way to use it in computations. They also allow one to define and calculate the antisymmetrized second derivatives, e.g., $\boldsymbol{d}^2\boldsymbol{v}$. The relation

$$\boldsymbol{d}^2\boldsymbol{v} = \mathcal{R}\boldsymbol{v}$$

where $\boldsymbol{v}$ is a vector will then introduce the "operator-valued" or "$\binom{1}{1}$-tensor valued" curvature 2-form $\mathcal{R}$. The notation of the extended exterior derivative puts a new look on the old apparatus of base vectors and parallel transport, and opens a way to calculate the curvature 2-form $\mathcal{R}$.

Let the vector field $\boldsymbol{v}$ be expanded in terms of some field of basis vectors $\boldsymbol{e}_\mu$; thus

$$\boldsymbol{v} = \boldsymbol{e}_\mu v^\mu.$$

Then the exterior derivative of this vector is

$$\boldsymbol{dv} = \boldsymbol{de}_\mu v^\mu + \boldsymbol{e}_\mu\, \boldsymbol{d}v^\mu.$$

Expand the typical vector-valued 1-form $\boldsymbol{de}_\mu$ in the form

$$\boldsymbol{de}_\mu = \boldsymbol{e}_\nu \boldsymbol{\omega}^\nu{}_\mu. \tag{14.14}$$

Here the "components" $\boldsymbol{\omega}^\nu{}_\mu$ in the expansion of $\boldsymbol{de}_\mu$ are 1-forms. Recall from equation (10.13) that the typical $\boldsymbol{\omega}^\nu{}_\mu$ is related to the connection coefficients by

$$\boldsymbol{\omega}^\nu{}_\mu = \Gamma^\nu{}_{\mu\lambda} \boldsymbol{\omega}^\lambda. \tag{14.15}$$

Therefore the expansion of the "vector" (really, "vector-valued 1-form") is

$$\boldsymbol{dv} = \boldsymbol{e}_\mu(\boldsymbol{d}v^\mu + \boldsymbol{\omega}^\mu{}_\nu v^\nu). \tag{14.16}$$

Now differentiate once again to find

$$\begin{aligned}
\boldsymbol{d}^2\boldsymbol{v} &= \boldsymbol{de}_\alpha \wedge (\boldsymbol{d}v^\alpha + \boldsymbol{\omega}^\alpha{}_\nu v^\nu) \\
&\quad + \boldsymbol{e}_\mu(\boldsymbol{d}^2 v^\mu + \boldsymbol{d\omega}^\mu{}_\nu v^\nu - \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{d}v^\nu) \\
&= \boldsymbol{e}_\mu(\boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{d}v^\alpha + \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^\alpha{}_\nu v^\nu \\
&\quad + \boldsymbol{d}^2 v^\mu + \boldsymbol{d\omega}^\mu{}_\nu v^\nu - \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{d}v^\alpha).
\end{aligned}$$

The simplifications made here use (1) the equation (14.14), for a second time; and (2) the product rule (14.13a), which introduced the minus sign in the last term, ready