Analysis of a Complex Kind

Week 5

Lecture 3: The Fundamental Theorem of Calculus for Analytic Functions

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Antiderivatives and Primitives

Recall the following conclusion of the fundamental theorem:

Fact

If $f : [a, b] \to \mathbb{R}$ is continuous and has an antiderivative $F : [a, b] \to \mathbb{R}$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Is there a complex equivalent?

Definition

Let $D \subset \mathbb{C}$ be a domain, and let $f : D \to \mathbb{C}$ be a continuous function. A primitive of f on D is an analytic function $F : D \to \mathbb{C}$ such that $F' = f$ on D.

An analytic function $F : D \to \mathbb{C}$ such that $F' = f$ is a primitive of f in D.

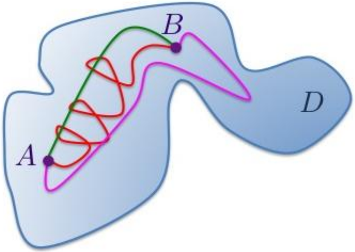
Theorem

If f is continuous on a domain D and if f has a primitive F in D, then for any curve $\gamma : [a, b] \to D$ we have that

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Notes:

- The integral only depends on the initial point and the terminal point of γ!
- Big 'hidden' assumption: f needs to have a primitive in D!
- Under what assumptions does f have a primitive?



Examples

1. Let $\gamma : [a, b] \to \mathbb{C}$ be the line segment from 0 to $1 + i$. What is $\int_\gamma z^2 dz$?

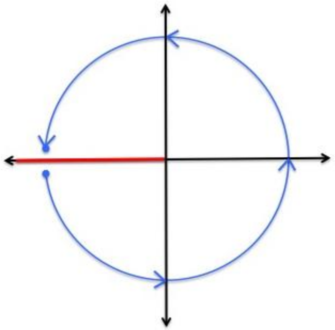
   The function $f(z) = z^2$ has a primitive in $\mathbb{C}$, namely $F(z) = \frac{1}{3} z^3$. Therefore,

$$\begin{aligned}
\int_0^{1+i} z^2 dz &= \int_\gamma f(z) dz = F(\gamma(b)) - F(\gamma(a)) \\
&= F(1+i) - F(0) \\
&= \frac{1}{3}(1+i)^3 - 0 \\
&= \frac{1}{3}(1 + 3i - 3 - i) = \frac{2}{3}(-1 + i).
\end{aligned}$$

Examples

2. Can we evaluate $\int_{|z|=1} \frac{1}{z} dz$ using a primitive?

The function $F(z) = \operatorname{Log} z$ satisfies that $F'(z) = \frac{1}{z}$, but not in *all* of $\mathbb{C}$ ...



- F is analytic in $\mathbb{C} \setminus (-\infty, 0]$.
- Let $\tilde{\gamma} : [a, b] \to \mathbb{C}$ be the part of the unit circle, started just below the negative x-axis, to just above the negative x-axis.
- Then

$$\begin{aligned} \int_\gamma \frac{1}{z} dz \approx \int_{\tilde{\gamma}} \frac{1}{z} dz &= \operatorname{Log}(\tilde{\gamma}(b)) - \operatorname{Log}(\tilde{\gamma}(a)) \\ &\approx \pi i - (-\pi i) = 2\pi i. \end{aligned}$$

Examples

3. Let γ be any curve in $\mathbb{C}$ from i to $\frac{i}{2}$. Then

$$\int_\gamma e^{\pi z} dz = \frac{1}{\pi} e^{\pi z} \Big|_i^{i/2} = \frac{1}{\pi} e^{\pi i/2} - \frac{1}{\pi} e^{\pi i} = \frac{1}{\pi}(i+1).$$

Examples

4. Let γ be any path in $\mathbb{C}$ from $-\pi i$ to πi. Then

$$\int_\gamma \cos z\,dz = \sin z|_{-\pi i}^{\pi i} = \sin(\pi i) - \sin(-\pi i).$$

But

$$\begin{aligned} \sin(\pi i) - \sin(-\pi i) &= \sin(\pi i) + \sin(\pi i) \\ &= 2\sin(\pi i) \\ &= 2\frac{e^{i\pi i} - e^{-i\pi i}}{2i} \\ &= -i(e^{-\pi} - e^{\pi}) = i(e^{\pi} - e^{-\pi}). \end{aligned}$$

When Does f Have a Primitive?

Theorem (Goursat)

Let D be a simply connected domain in $\mathbb{C}$, and let f be analytic in D. Then f has a primitive in D. Moreover, a primitive is given explicitly by picking $z_0 \in D$ and letting

$$F(z) = \int_{z_0}^{z} f(w)dw,$$

where the integral is taken over an arbitrary curve in D from z_0 to z.

One way to prove this theorem is as follows:

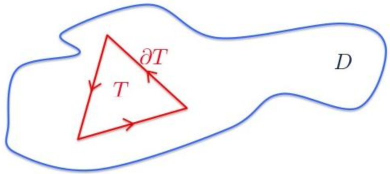
1. First, show *Morera's Theorem*: If f is continuous on a simply connected domain D, and if $\int_\gamma f(z)dz = 0$ for any triangular curve γ in D, then f has a primitive in D.
2. Next, show the *Cauchy Theorem for Triangles*: For any triangle T that fits into D (including its boundary), $\int_{\partial T} f(z)dz = 0$.

Theorem (Cauchy for Triangles)

Let D be an open set in $\mathbb{C}$, and let f be analytic in D. Let T be a triangle that fits into D (including its boundary), and let ∂T be its boundary, oriented positively. Then

$$\int_{\partial T} f(z)dz = 0.$$

Proof idea:



Subdivide the triangle into four equal-sized triangles.

- The integral of f over ∂T is the same as the sum of the four integrals over the boundaries of the smaller triangles!
- Use the *ML*-estimate and delicate balancing of boundary length of triangles and the fact that

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \varepsilon(z - z_0)$$

for z near a point z_0 inside T.

Morera's Theorem

Theorem (Morera)

If f is continuous on a simply connected domain D, and if $\int_\gamma f(z)dz = 0$ for any triangular curve in D, then f has a primitive in D.

Proof idea:

- First, show Morera's theorem in a disk (the proof is not hard and resembles the proof of the real-valued fundamental theorem of calculus).
- Extending the result to arbitrary simply connected domains is not that easy. This part of the proof requires the use of Cauchy's Theorem for simply connected domains. We'll discuss this theorem in the next lecture.