Theorem 4.1 *Given the functional confluent Vandermonde matrix* (7)–(9) *with incidence vector* $\mathcal{V}$ *wanting "stars", the unique LU-factorization with unitary diagonal elements* $L_{i,i} = 1$ *is given by the formulae*:

$$\begin{cases} L_{i,1} = x_1^{i-1} & \text{for } 1 \le i \le \delta, \\ U_{1,j} = \Upsilon_{1,j} & \text{for } 1 \le j \le \delta, \\ L_{i,j} = L_{i-1,j-1} + L_{i-1,j}\xi_j & \text{for } 2 \le j \le i, \\ U_{i,j} = \big(\varkappa(j) - 1\big)U_{i-1,j-1} + U_{i-1,j}(x_{\varrho(j)} - \xi_{i-1}) & \text{for } 2 \le i \le j. \end{cases} \tag{20}$$

Proof of Theorem 4.1 Define a matrix Ω by:

$$\Omega_{i,j} = \sum_{k=1}^{D} L_{i,k}U_{k,j} = \sum_{k=1}^{i} L_{i,k}U_{k,j} \quad 1 \le i, j \le \delta. \tag{21}$$

In what follows, we prove that $\Omega_{i,j} = \Upsilon_{i,j}$ $\forall (i, j)$ $1 \le i, j \le \delta$. The proof is a total 2D recurrence-based that can be summarized as follows:

- Initialization by proving:
  - $\Omega_{i,j} = \Upsilon_{i,j}$ for $i = 1$ and $1 \le j \le \delta$ that is $\Upsilon_{1,j}$.
  - $\Omega_{i,j} = \Upsilon_{i,j}$ for $j = 1$ and $1 \le i \le \delta$ that is $\Upsilon_{i,1}$.
- Assuming $\Omega_{i,j} = \Upsilon_{i,j}$ holds for any $1 \le i \le i_0 - 1$ and $1 \le j \le j_0 - 1$ and proving:
  - $\Omega_{i_0,j_0-1} = \Upsilon_{i_0,j_0-1}$.
  - $\Omega_{i_0-1,j_0} = \Upsilon_{i_0-1,j_0}$.
  - $\Omega_{i_0,j_0} = \Upsilon_{i_0,j_0}$.
- Conclude that $\Omega_{i,j} = \Upsilon_{i,j}$ for $1 \le i \le \delta$ and $1 \le j \le \delta$.

Since L is a lower triangular matrix with a unitary diagonal and U is an upper triangular, using (20) one proves $\Omega_{1,j} = U_{1,j} \equiv \Upsilon_{1,j}$ and $\Omega_{i,1} = L_{i,1}U_{1,1} \equiv \Upsilon_{i,1}$ for any $1 \le j \le \delta$ and any $1 \le i \le \delta$. Hence, the initialization assumption holds. Assume now that $\Omega_{i,j} = \Upsilon_{i,j}$ is satisfied for any $1 \le i \le i_0 - 1$ and $1 \le j \le j_0 - 1$. According to (20), one gets:

$$\begin{cases} L_{i_0,k} = L_{i_0-1,k-1} + L_{i_0-1,k}\xi_k, \\ U_{k,j_0-1} = \big(\varkappa(j_0 - 1) - 1\big)U_{k-1,j_0-2} + U_{k-1,j_0-1}(x_{\varrho(j_0-1)} - \xi_{k-1}), \end{cases}$$

then

$$\begin{aligned} L_{i_0,k}U_{k,j_0-1} = {} & \big(\varkappa(j_0 - 1) - 1\big)L_{i_0-1,k-1}U_{k-1,j_0-2} + x_{\varrho(j_0-1)}L_{i_0-1,k-1}U_{k-1,j_0-1} \\ & - \xi_{k-1}L_{i_0-1,k-1}U_{k-1,j_0-1} + \xi_k L_{i_0-1,k}U_{k,j_0-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \Omega_{i_0,j_0-1} &= x_{\varrho(j_0-1)} \sum_{k=1}^{i_0} L_{i_0-1,k}U_{k,j_0-1} + \sum_{k=1}^{i_0} \big(\varkappa(j_0 - 1) - 1\big)L_{i_0-1,k}U_{k,j_0-2} \\ &= x_{\varrho(j_0-1)}\Upsilon_{i_0-1,j_0-1} + \big(\varkappa(j_0 - 1) - 1\big)\Upsilon_{i_0-1,j_0-2} \triangleq \Upsilon_{i_0,j_0-1}. \end{aligned}$$

The same argument gives

$$\begin{cases} L_{i_0-1,k} = L_{i_0-2,k-1} + L_{i_0-2,k}\xi_k, \\ U_{k,j_0} = \big(\varkappa(j_0)-1\big)U_{k-1,j_0-1} + U_{k-1,j_0}(x_{\varrho(j_0)} - \xi_{k-1}), \end{cases}$$

then

$$L_{i_0-1,k}U_{k,j_0} = x_{\varrho(j_0)}L_{i_0-2,k-1}U_{k-1,j_0} + \big(\varkappa(j_0)-1\big)L_{i_0-2,k-1}U_{k-1,j_0-1} - \xi_{k-1}L_{i_0-2,k-1}U_{k-1,j_0} + \xi_k L_{i_0-2,k}U_{k,j_0}.$$

Thus,

$$\begin{aligned} \Omega_{i_0-1,j_0} &= x_{\varrho(j_0)}\sum_{k=1}^{i_0} L_{i_0-2,k}U_{k,j_0} + \sum_{k=1}^{i_0}\big(\varkappa(j_0)-1\big)L_{i_0-2,k}U_{k,j_0-1} \\ &= x_{\varrho(j_0)}\Upsilon_{i_0-2,j_0} + \big(\varkappa(j_0)-1\big)\Upsilon_{i_0-2,j_0-1} \triangleq \Upsilon_{i_0-1,j_0}. \end{aligned}$$

By using again (20) one obtains:

$$\begin{cases} L_{i_0,k} = L_{i_0-1,k-1} + L_{i_0-1,k}\xi_k, \\ U_{k,j_0} = \big(\varkappa(j_0)-1\big)U_{k-1,j_0-1} + U_{k-1,j_0}(x_{\varrho(j_0)} - \xi_{k-1}), \end{cases}$$

leading to:

$$L_{i_0,k}U_{k,j_0} = x_{\varrho(j_0)}L_{i_0-1,k-1}U_{k-1,j_0} + \big(\varkappa(j_0)-1\big)L_{i_0-1,k-1}U_{k-1,j_0-1} + \xi_k L_{i_0-1,k}U_{k,j_0} - \xi_{k-1}L_{i_0-1,k-1}U_{k-1,j_0}.$$

Hence, we have:

$$\Omega_{i_0,j_0} = x_{\varrho(j_0)}\Upsilon_{i_0-1,j_0} + \big(\varkappa(j_0)-1\big)\Upsilon_{i_0-1,j_0-1} \triangleq \Upsilon_{i_0,j_0},$$

which ends the proof. □