Non linear differential quation

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0.1 Introduction

This essay deals with 1st order non linear differential equations without second member. Here is the equation :

$$ay' + by^n = 0, a, b, n \in R \tag{1}$$

It's known that for every 1st order ODE without second member, the solutions are exprimed by an exponentional function like $y = Ae^{kx}$.
We can try this approach for our problem.

0.2 Resolution

Then we have :

$$ay' + by^n = 0 \tag{2}$$

$$a(Ae^{kx})' + b(Ae^{kx})^n = 0 \tag{3}$$

Remind the derivate of the function e^x is particuliar : $(e^x)' = e^x$ Applying the derivate's laws and after simplification, we have :

$$ke^kx + \frac{b}{a}A^{n-1}e^{nkx} = 0 \tag{4}$$

In order to clarify the next calculations, we set $r = e^{kx}$:

$$kr + \frac{b}{a}A^{n-1}r^n = 0 \tag{5}$$

We can factorize yet :

$$r(k + \frac{b}{a}A^{n-1}r^{n-1}) = 0 \tag{6}$$

A product of two factors is zero if and only if at least one of the two factors is zero. However, r can not be zero here.
Indeed, if $r = 0$, then $e^{kx} = 0$ or also $k = \frac{\ln(0)}{x}$ but $\ln(0)$ does not exist, so $r \neq 0$ and the second part of this product is necessarily zero.
Then the equation become :

$$k + \frac{b}{a}A^{n-1}r^{n-1} = 0 \tag{7}$$

And we can reintroduce $e^{kx} = r$:

$$k + \frac{b}{a}A^{n-1}e^{(n-1)kx} = 0 \tag{8}$$

Here is an equation solvable via the Lambert function, denoted $\mathrm{W}(x)$.
For the next calculations, we set :
- $xe^x = y \rightarrow \mathrm{W}(y) = x$

- $\lambda = \frac{b}{a} A^{n-1}$
So :

$$k + \lambda e^{(n-1)kx} = 0 \tag{9}$$

After some rearrangements, we obtain an issue to use the Lambert function :

$$\lambda e^{(n-1)kx} = -k \tag{10}$$

$$\lambda = \frac{-k}{e^{(n-1)kx}} \tag{11}$$

$$\lambda = -ke^{-(n-1)kx} \tag{12}$$

$$\lambda(n-1)x = -(n-1)kxe^{-(n-1)kx} \tag{13}$$

Remind we don't care about x, but about k.
Then we find :

$$\mathrm{W}\left(\lambda(n-1)x\right) = -(n-1)kx \tag{14}$$

It is now enough to isolate k :

$$k = \frac{\mathrm{W}\left(\lambda(n-1)x\right)}{-(n-1)x} \tag{15}$$

Finally :

$$k = \frac{\mathrm{W}\left(\frac{b}{a}A^{n-1}(n-1)x\right)}{-(n-1)x} \tag{16}$$

We succeed to determine k, but a problem survive : k depends on A.
$y(x)$ does not look like the exponentional function Ae^{kx}. To solve the problem, it is assumed that $A = 1$.
So $y = e^{kx}$ and k become :

$$k = \frac{\mathrm{W}\left(\frac{b}{a}(n-1)x\right)}{-(n-1)x} \tag{17}$$

We can introduce k in the function's expression, and we notice that x can be simplified, x supposed different than zero.

$$y(x) = e^{-\frac{\mathrm{W}\left(\frac{b}{a}(n-1)x\right)}{n-1}} \tag{18}$$