Other Groups in $\mathcal{M}_n(\mathbb{K})$ and $\mathcal{M}_2(\mathbb{K})$

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1 Introduction

2 Notations

Let D be an idempotent matrix, that is $D \in M_n(\mathbb{K})$ where $\mathbb{K}$ is a commutative field of 0 charactristic, and $D^2 = D$.

Let $GL_n(\mathbb{K})$ be the set of invertible matrices of $M_n(\mathbb{K})$.

Let $\mathcal{G}_D = \{A \cdot D \mid A \in GL_n(\mathbb{K}) \wedge (A \cdot D = D \cdot A)\}$

Let I_n the identity matrix of $M_n(\mathbb{K})$, and 0_n the null matrix.

Let $\mathcal{P}_0$ be the set of 2 nilpotent matrices of $M_2(\mathbb{K})$, and $\mathcal{P}_0^\star = \mathcal{P}_0 \setminus \{0_2\}$

Let $\mathcal{D}_1$ bet the set of idempotent matrices, and $\mathcal{D}_1^\star$ the set of non trivial idempotent matrices

3 Some results in $M_n(\mathbb{K})$

We will show that $(\mathcal{G}_D, \cdot, D)$ is a group whose identity is D :

1. $\mathcal{G}_D$ is closed under multiplication $\cdot$
2. $(\mathcal{G}_D, \cdot, D)$ is associative
3. D is the identity of $(\mathcal{G}_D, \cdot, D)$
4. Any matrix in $\mathcal{G}_D$ has an inverse in $\mathcal{G}_D$

Proofs :

1. Let $A \cdot D \in \mathcal{G}_D$ and $B \cdot D \in \mathcal{G}_D$ then

$$\begin{array}{rcll}
(A \cdot D) \cdot (B \cdot D) & = & A \cdot (D \cdot B) \cdot D & \text{The matrices multiplication is associative} \\
 & = & A \cdot (B \cdot D) \cdot D & \text{By definition of } \mathcal{G}_D \\
 & = & (A \cdot B) \cdot D^2 & \text{The matrices multiplication is associative} \\
 & = & (A \cdot B) \cdot D & \text{Definition of idempotence} \\
 & & & \\
 & = & (D \cdot A) \cdot (D \cdot B) & \text{By definition of } \mathcal{G}_D \\
 & = & D \cdot (A \cdot D) \cdot B & \text{The matrices multiplication is associative} \\
 & = & D \cdot (D \cdot A) \cdot B & \text{By definition of } \mathcal{G}_D \\
 & = & D^2 \cdot (A \cdot B) & \text{The matrices multiplication is associative} \\
 & = & D \cdot (A \cdot B) & \text{Definition of idempotence}
\end{array}$$

   The product of invertible matrices being invertible : QED.

2. The product of matrices is associative.
3. Let $A \cdot D \in \mathcal{G}_D$

$$\begin{array}{rcll}
(A \cdot D) \cdot D & = & A \cdot (D \cdot D) & \text{The matrices multiplication is associative} \\
 & = & A \cdot D & \text{By definition of idempotence} \\
D \cdot (A \cdot D) & = & D \cdot (D \cdot A) & \text{By definition of } \mathcal{G}_D \\
 & = & (D \cdot D) \cdot A & \text{The matrices multiplication is associative} \\
 & = & D \cdot A & \text{By definition of idempotence} \\
 & = & A \cdot D & \text{By definition of } \mathcal{G}_D
\end{array}$$

   QED

4. Let $A \cdot D \in \mathcal{G}_D$, A being invertible, it's inverse (within the meaning of $GL_n(\mathbb{K})$), A^{-1}, exists :

   If A commutes with D then A^{-1} commutes with D (By multiplying both sides of $A \cdot D = D \cdot A$ by A^{-1} on the left and on the right we get $D \cdot A^{-1} = A^{-1} \cdot D$)

$$\begin{array}{rcll}(A\cdot D)\cdot(A^{-1}\cdot D) & = & (A\cdot A^{-1})\cdot(D\cdot D) & \text{Properties of } \cdot \text{ and } \mathcal{G}_D \\ & = & I_n\cdot D & \text{Definition of idempotence and inverse} \\ & = & D & \text{Definition of the Identity of } M_n(\mathbb{K}) \\ (A^{-1}\cdot D)\cdot(A\cdot D) & = & D & \text{Same demonstraion}\end{array}$$

QED

4 Some results in $M_2(\mathbb{K})$

4.1 How does look an idempotent 2x2 matrix

Let $\begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}$ be an idempotent matrix :

$$\begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}^2 = \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}$$

$$\begin{pmatrix}\alpha^2+\gamma\beta & \gamma(\alpha+\delta) \\ \beta(\alpha+\delta) & \gamma\beta+\delta^2\end{pmatrix} = \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix} \Leftrightarrow \begin{cases}\alpha^2+\gamma\beta & = & \alpha \\ \gamma(\alpha+\delta) & = & \gamma \\ \beta(\alpha+\delta) & = & \beta \\ \gamma\beta+\delta^2 & = & \delta\end{cases}$$

Either : $\alpha+\delta \neq 1$ then $(\gamma = 0)\wedge(\beta = 0)\wedge(\alpha^2 = \alpha)\wedge(\delta^2 = \delta)$ that is to say that α and δ are the idempotents of $\mathbb{K}$, 0 an 1, so we get the two (trivial) idempotent matrices : $\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = I_2$ and $\begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix} = 0_2$. We may notice that $\mathcal{G}_{I_2} = GL_2(\mathbb{K})$ and $\mathcal{G}_{0_2} = \{0_2\}$.

Or : $\alpha+\delta = 1$ then $(\alpha^2+\gamma\beta = \alpha)\wedge(\gamma\beta+\delta^2 = \delta)$
$\Leftrightarrow$
$(\alpha^2+\gamma\beta = \alpha(\alpha+\delta)\wedge(\gamma\beta+\delta^2 = \delta(\alpha+\delta))$
$\Leftrightarrow$
$\gamma\beta = \alpha\delta$

So, an idempotent matrix of $D \in M_2(\mathbb{K})$ is either a trivial one 0_2 or I_2, or a non invertible matrix whose trace is 1 : $D = \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}$, such as $\begin{cases}\alpha+\delta & = & 1 \\ \alpha\delta & = & \gamma\beta\end{cases}$

4.2 How does look a 2-nilpotent 2x2 matrix

Let $X = \begin{pmatrix}x & z \\ y & t\end{pmatrix}$ be a 2-nilpotent matrix :

$$\begin{pmatrix}x & z \\ y & t\end{pmatrix}^2 = \begin{pmatrix}x^2+yz & z(x+t) \\ y(x+t) & t^2+yz\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$$

As X is not invertible, we have $xt = yz$, which gives : $\begin{pmatrix}x & z \\ y & t\end{pmatrix}^2 = \begin{pmatrix}x(x+t) & z(x+t) \\ y(x+t) & t(x+t)\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$ whose only solution is $(x+t) = 0$

Theorem : A non invertible 2x2 matrix is nilpotent if and only if it's trace is null.

4.3 Do the $\mathcal{G}_D$'s cover $M_2(\mathbb{K})$?

Let $X \in M_2(\mathbb{K})$, can we find an idempotent matrix D and an invertible matrix A such as $X = D\cdot A = A\cdot D$?

If X is invertible, the only solution is $D = I_2$ and $A = X$ so $X \in \mathcal{G}_X$ (I_2 is the only invertible idempotent matrix).

If $X = 0_2$, the only solution is $D = 0_2$ and A is any invertible matrix, so $0_2 \in \mathcal{G}_{0_2}$.

If X is a not null, 2-nilpotent matrix (like $\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$, for instance)
$X = D\cdot A \Rightarrow X^2 = (D\cdot A)\cdot(D\cdot A) = D^2\cdot A^2 = D\cdot A^2 = 0_2$, but, as A is invertible, this implies $D = 0_2$ and so $X = 0_2$, therefore, it is not possible to find D such as $X \in \mathcal{G}_D$.

If X is a non 2-nilpotent (so $D \neq 0_2$), non invertible matrix, $X = D \cdot A \Rightarrow D \cdot X = D^2 \cdot A = D \cdot A = X$ and $X = A \cdot D \Rightarrow X \cdot D = A \cdot D^2 = A \cdot D = X$

Let $X = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$ as X is non invertible $xt = yz$ and as it is not a 2-nilpotent matrix, $(x+t) \neq 0$

Let $D = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, as D is idempotent and not in $\{I_2, 0_2\}$ then we have $\alpha\delta = \beta\gamma$ and $\alpha + \delta = 1$

$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \cdot \begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \cdot \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$

❶ $\alpha x + \gamma y = x$	❺ $\alpha x + \beta z = x$
❷ $\beta x + \delta y = y$	❻ $\alpha y + \beta t = y$
❸ $\alpha z + \gamma t = z$	❼ $\gamma x + \delta z = z$
❹ $\beta z + \delta t = t$	❽ $\gamma y + \delta t = t$

If we add equation ❷ and ❻ we get $\beta(x+t) + (\alpha+\delta)y = 2y$, or, as $\alpha + \delta = 1$ and $x + t \neq 0 : \beta = \dfrac{y}{x+t}$

If we add equation ❸ and ❼ we get $\gamma(x+t) + (\alpha+\delta)z = 2z$, or, as $\alpha + \delta = 1$ and $x + t \neq 0 : \gamma = \dfrac{z}{x+t}$

From equation ❶ and the previouly calculated value of γ, we get : $\alpha x + \dfrac{yz}{x+t} = t$, as $yz = xt$, $\alpha x + \dfrac{xt}{x+t} = t$ or $\alpha x = \dfrac{x^2}{x+t}$.

From equation ❽ and $yz = xt$, we get (with the same method as above) $\delta t = \dfrac{t^2}{x+t}$

Either : $x \neq 0$ then $\alpha = \dfrac{x}{x+t}$ and as ($\alpha + \delta = 1$, we get $\delta = \dfrac{t}{t+t}$).

Or : $x = 0$, so, as $x + t \neq 0$ then ($t \neq 0$) from which we can infer : $\delta = \dfrac{t}{x+t}$ and as ($\alpha + \delta = 1$, we get $\alpha = \dfrac{x}{t+t}$).

So the $(\mathcal{G}_D, \cdot, D)$ are disjoint groups, we will show that they cover $M_2(\mathbb{K})$ with the only exception of the not null, 2 nilpotent matrices.

1. X invertible $\Rightarrow X \in \mathcal{G}_X$
2. $0_2 \in \mathcal{G}_{0_2}$
3. $X \notin (GL_n(\mathbb{K}) \cup \{0_2\} \cup \mathcal{P}_0) \Rightarrow X = \frac{1}{Tr(X)} X \cdot Tr(X) I_2$, so $X \in \mathcal{G}_{\frac{1}{Tr(X)}X}$

Theorem : $M_2(\mathbb{K}) = \mathcal{P}_0^\star \bigcup_{D \in \mathcal{D}_1} \mathcal{G}_D$, and it is a non overlapping covering of $M_2(\mathbb{K})$.

We can also write : $M_2(\mathbb{K}) = \mathcal{P}_0^\star \cup \{0_2\} \cup GL_2(\mathbb{K}) \bigcup_{D \in \mathcal{D}_1^\star} \mathcal{G}_D$