CHAPTER 11

BENDING OF ANISOTROPIC PLATES

85. Differential Equation of the Bent Plate. In our previous discussions we have assumed that the elastic properties of the material of the plate are the same in all directions. There are, however, cases in which an anisotropic material must be assumed if we wish to bring the theory of plates into agreement with experiments.[1] Let us assume that the material of the plate has three planes of symmetry with respect to its elastic properties.[2] Taking these planes as the coordinate planes, the relations between the stress and strain components for the case of plane stress in the xy plane can be represented by the following equations:

$$\begin{aligned} \sigma_x &= E'_x\epsilon_x + E''\epsilon_y \\ \sigma_y &= E'_y\epsilon_y + E''\epsilon_x \\ \tau_{xy} &= G\gamma_{xy} \end{aligned} \tag{a}$$

It is seen that in the case of plane stress, four constants, E'_x, E'_y, E'', and G, are needed to characterize the elastic properties of a material.

Considering the bending of a plate made of such a material, we assume, as before, that linear elements perpendicular to the middle plane (xy plane) of the plate before bending remain straight and normal to the deflection surface of the plate after bending.[3] Hence we can use our previous expressions for the components of strain:

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y} \tag{b}$$

[1] The case of a plate of anisotropic material was discussed by J. Boussinesq, *J. math.*, ser. 3, vol. 5, 1879. See also Saint Venant's translation of "Théorie de l'élasticité des corps solides," by A. Clebsch, note 73, p. 693.

[2] Such plates sometimes are called "orthotropic." The bending of plates with more general elastic properties has been considered by S. G. Lechnitzky in his book "Anisotropic Plates," 2d ed., Moscow, 1957.

[3] The effect of transverse shear in the case of anisotropy has been considered by K. Girkmann and R. Beer, *Österr. Ingr.-Arch.*, vol. 12, p. 101, 1958.

The corresponding stress components, from Eqs. (a), are

$$\begin{aligned} \sigma_x &= -z\left(E'_x\frac{\partial^2 w}{\partial x^2} + E''\frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y &= -z\left(E'_y\frac{\partial^2 w}{\partial y^2} + E''\frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy} &= -2Gz\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned} \tag{c}$$

With these expressions for stress components the bending and twisting moments are

$$\begin{aligned} M_x &= \int_{-h/2}^{h/2}\sigma_x z\,dz = -\left(D_x\frac{\partial^2 w}{\partial x^2} + D_1\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= \int_{-h/2}^{h/2}\sigma_y z\,dz = -\left(D_y\frac{\partial^2 w}{\partial y^2} + D_1\frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -\int_{-h/2}^{h/2}\tau_{xy} z\,dz = 2D_{xy}\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned} \tag{212}$$

in which

$$D_x = \frac{E'_x h^3}{12} \qquad D_y = \frac{E'_y h^3}{12} \qquad D_1 = \frac{E'' h^3}{12} \qquad D_{xy} = \frac{G h^3}{12} \tag{d}$$

Substituting expressions (212) in the differential equation of equilibrium (100), we obtain the following equation for anisotropic plates:

$$D_x\frac{\partial^4 w}{\partial x^4} + 2(D_1 + 2D_{xy})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = q$$

Introducing the notation

$$H = D_1 + 2D_{xy} \tag{e}$$

we obtain

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = q \tag{213}$$

The corresponding expressions for the shearing forces are readily obtained from the conditions of equilibrium of an element of the plate (Fig. 48) and the previous expressions for the moments. Thus, we have

$$\begin{aligned} Q_x &= -\frac{\partial}{\partial x}\left(D_x\frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right) \\ Q_y &= -\frac{\partial}{\partial y}\left(D_y\frac{\partial^2 w}{\partial y^2} + H\frac{\partial^2 w}{\partial x^2}\right) \end{aligned} \tag{214}$$

In the particular case of isotropy we have

$$E'_x = E'_y = \frac{E}{1-\nu^2} \qquad E'' = \frac{\nu E}{1-\nu^2} \qquad G = \frac{E}{2(1+\nu)}$$