

# *A Nonlinear Quantum Theory With Non Local And Non Commutative Hidden Variables*

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## **Abstract**

We modify in this paper the Schrödinger equation, as a result of the delinearization non local and non commutative hidden variables are added to quantum mechanics.

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*To Tarek, Jade & Natacha*

## **1) Introduction**

The success of standard quantum mechanics (QM) for decades and in many fields of physics is indisputable. However, we believe that numbers of paradoxes and many problems engendered by the conventionnal theory motivate the search for alternate issues. More investigation has to be done on the foundations which must have to undergo a profound revision. The first physicists to suggest that QM is incomplete was Einstein and De Broglie<sup>[1]</sup>. Since then more and more people of the community are attempting to find the correct modifications <sup>[2]</sup>. It is a worthwhile task to achieve the goal of constructing a complete and satisfactory non-local hidden variables theory ,

but as we all know it is not easy to fulfil such a program. The main efforts made in this paper are based on Einstein idea, which consist in introducing in a suitable fashion the sources (and sinks) of the field inside the evolution equation. Our modification of Schrödinger equation has for natural consequence, as we will see in the process of delinearization, that non local and non commutative variables are automatically added. Unfortunately the theory we are left with is not background independant. This lack of background independance is a weakness of our approach. The plan of our work is as follows. First we will show how to find, from the associated classical problem, the expression of the sources (and sinks). Next, in section 3, we manage to introduce new terms inside the classical Kamiltonian, finally we proceed to the quantization of the theory.

Let's start from standard QM, Schrödinger equation is

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \mathbf{H}_q(t) |\Psi(t)\rangle \quad (1)$$

+ *Boundary conditions, Cauchy's data...*

with

$$\frac{d}{dt} \langle \Psi(t) | \Psi(t) \rangle = 0 \quad (2)$$

In the continuous base  $\{|\mathbf{r}_1, \dots, \mathbf{r}_n\rangle\}$  equation (1) read

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t) = \mathbf{H}_0 \psi(\mathbf{r}_1, \dots, \mathbf{r}_n; t) \quad (3)$$

In this work we are going to consider that this linear equation is wrong. We shall simply reject the validity of (1) while keeping most principles and postulates of QM.

## 2) Taking account of sources and sinks

### 2-a) Conservation relation

Let's introduce the closure relation  $\int \cdots \int_{\mathcal{L}} |\mathbf{r}_1, \dots, \mathbf{r}_n\rangle \langle \mathbf{r}_1, \dots, \mathbf{r}_n| d^3r_1 \cdots d^3r_n =$

1 in (2), we get

$$\frac{d}{dt} \int_{\mathcal{L}} \cdots \int_{\mathcal{L}} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 d^3r_1 \dots d^3r_n = 0 \quad (4)$$

This means that we can interpret (4) according to the associated classical problem by saying that the integral in (4) is a constant of the movement. On the other hand we have the continuity equation

$$\int_{\mathcal{L}} \cdots \int_{\mathcal{L}} \frac{\partial}{\partial t} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 d^3r_1 \dots d^3r_n + \int_{\mathcal{L}} \cdots \int_{\mathcal{L}} \vec{\nabla}_{\psi} \cdot \vec{J}_{\psi} d^3r_1 \dots d^3r_n = 0 \quad (5)$$

where  $\vec{\nabla}_{\psi} \cdot \vec{J}_{\psi}$  is for the  $n$  particles fluid

$$\vec{\nabla}_{\psi} \cdot \vec{J}_{\psi} := \sum_{k=1}^n \vec{\nabla}_k \cdot \left( |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k \right) \quad (6)$$

where  $\vec{V}_k$  being the speed vector for particle  $k$ . That is, for any boundary  $\mathcal{L}$ , the measures  $d^3r_1 \dots d^3r_n$  being continuous and positive we can write

$$\frac{\partial}{\partial t} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 + \sum_{k=1}^n \vec{\nabla}_k \cdot \left( |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k \right) = 0 \quad (7)$$

In the associated classical problem, the classical hamiltonian  $H_{0 \text{ class}}(\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n, t)$  being the one from which the quantum hamiltonian  $\mathbf{H}_0$  appearing in (3) is constructed, the density function must then satisfy to

$$\frac{d}{dt} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 - \left\{ H_{0 \text{ class}}(\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n, t), |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \right\}_{\mathbf{r}, \mathbf{p}} = \frac{\partial}{\partial t} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \quad (8)$$

So, if we put  $\sigma_{\psi} := \frac{\partial}{\partial t} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 + \sum_{k=1}^n \vec{\nabla}_k \cdot \left( |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k \right)$  as sources (and sinks) terms, then (7) becomes

$$\sigma_\psi := \frac{d}{dt} |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 - \left\{ H_{0 \text{ class}}(\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n, t), |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \right\}_{\mathbf{r}, \mathbf{p}} + \sum_{k=1}^n \vec{\nabla}_k \cdot \left( |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k \right) = 0 \quad (9)$$

What we can say now is that simply in the case of the Schrödinger equation the sources (and sinks) terms are null, so to generalise the Schrödinger equation we must take account of the case in which those terms are not zero. Now how to introduce them inside the evolution equation? In the next sections we will see how to overcome this difficulty.

### 2-b) Kamiltonian with sources and sinks

Our classical hamiltonian  $H_{0 \text{ class}}$  from which the quantum equation (3) is constructed, is defined with the use of the lagrangian as

$$\begin{aligned} H_{0 \text{ class}}(\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{p}_1, \dots, \mathbf{p}_n, t) &= \sum_{\alpha=1}^n \mathbf{p}_\alpha \dot{\mathbf{r}}_\alpha - \\ L_{o \text{ class}}(\mathbf{r}_1, \dots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_n, t) & \end{aligned} \quad (10)$$

If we transform canonically  $H_{0 \text{ class}}$  with some change of the canonical coordinates, say

$$\begin{aligned} Q_\alpha &= \mu_\alpha (\mathbf{r}_\alpha + \delta \mathbf{r}_\alpha) \\ P_\alpha &= \nu_\alpha (\mathbf{p}_\alpha + \delta \mathbf{p}_\alpha) \end{aligned} \quad (11)$$

then according to Hamilton's principle the Kamiltonian

$$K_{\text{class}}(Q_\alpha, P_\alpha, t) = \sum_{\alpha=1}^n P_\alpha \dot{Q}_\alpha - L_{K_{\text{class}}}(Q_\alpha, \dot{Q}_\alpha, t), \quad (12)$$

is connected to  $H_{0 \text{ class}}$  by

$$\begin{aligned} \mu_\alpha \nu_\alpha \left( \sum_{\alpha=1}^n \mathbf{p}_\alpha \dot{\mathbf{r}}_\alpha - H_{0 \text{ class}}(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t) \right) &= \sum_{\alpha=1}^n P_\alpha \dot{Q}_\alpha - \\ K_{\text{class}}(Q_\alpha, P_\alpha, t) + \frac{d}{dt} F_{\mu_\alpha \nu_\alpha} & \end{aligned} \quad (13)$$

where  $\mu_\alpha$  and  $\nu_\alpha$  are two parameters of scale transformations (see<sup>[3]</sup>). Such transformations are known to be the "Extended Canonical Transformations". Since  $F_{\mu_\alpha \nu_\alpha}$  is an arbitrary function we shall assume it to be

$$F_{\mu_\alpha \nu_\alpha} = \mu_\alpha \nu_\alpha \left( \sum_{\alpha=1}^n \mathbf{r}_\alpha P_\alpha + \varepsilon G(\mathbf{r}_\alpha, P_\alpha, t) \right) \quad (14)$$

where  $G$  is again any generating function of infinitesimal canonical transformations. The infinitesimal parameter appearing in (14) is always  $\varepsilon \neq 0$ . As all calculations must be carried out up to first-order,  $G$  turns out to be a function of  $(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t)$  only and the infinitesimals are just the same as those given by Goldstein in his book

$$\begin{aligned} \delta \mathbf{p}_\alpha &= -\varepsilon \frac{\partial}{\partial \mathbf{r}_\alpha} G(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t), \\ \delta \mathbf{r}_\alpha &\simeq \varepsilon \frac{\partial}{\partial \mathbf{p}_\alpha} G(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t) \end{aligned} \quad (15)$$

By using these we can establish an expression for  $\frac{d}{dt} F_{\mu_\alpha \nu_\alpha}$  which makes us reach the general form of our kamiltonian

$$\begin{aligned} K_{class}(Q_\alpha, P_\alpha, t) &:= K_{class}(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t) = \\ \mu_\alpha \nu_\alpha &\left( H_{0 \ class}(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t) + \varepsilon \frac{\partial}{\partial t} G(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t) + O(\varepsilon^2) \right) \end{aligned} \quad (16)$$

Notice that if we need to find again our  $H_{0 \ class}$  of the beginning, with this E.I.C.T (for extended infinitesimal canonical transformation), we must set  $\mu_\alpha \nu_\alpha = 1$  and

$$\varepsilon \frac{\partial}{\partial t} G(\mathbf{r}_\alpha, \mathbf{p}_\alpha, t) + O(\varepsilon^2) = 0 \quad (17)$$

in (16), which is the same as applying the identity transformations to  $H_{0 \ class}$ .

### 3) Nonlinear equation of evolution

Now let's drop the  $\alpha$  subscript and return to the old variables. We first multiply equation (9) by **any integrable well-ordered function in the sense of Schrödinger** (see<sup>[4]</sup>), say  $\Phi(\mathbf{r}_1, \dots, \mathbf{r}_k, \mathbf{p}_1, \dots, \mathbf{p}_n, G, \hbar, c, k_b, e, t)$  with appropriate physical dimension (involving length only) for reason of homogeneity (here  $k \leq n$ ), and where  $G, \hbar, c$ , are the fundamental constants of nature and  $k_b$  the Boltzmann constant. We have also introduced the quantum of charge  $e$ , thus the mass of particles can be quantified with the help of  $e$ . Now let's define the function:  $\Phi_\varepsilon := \varepsilon \Phi(\mathbf{r}_1, \dots, \mathbf{r}_k, \mathbf{p}_1, \dots, \mathbf{p}_n, G, \hbar, c, k_b, e, t)$ , this function will represent a membrane in several dimensions, we will consider it as one of the parameters of our theory. If we suppose that we have

$\xi$  (with  $\xi \leq n$ ) point source in our system of particles, that are located at positions  $\{r_z, \dots, r_s\}$ , we then may integrate  $3\xi$  times over the domain  $\mathcal{L}$  (or over infinity) to avoid singular interaction (anomalies and divergencies) from appearing, and also over time, we will have

$$\int \cdots \int \Phi_\varepsilon \sigma_\psi d^3 r_z \dots d^3 r_s dt = 0 \quad (18)$$

If we don't represent any of the particle as a mathematical point there is no need to integrate and we may just leave the expression of  $\sigma_\psi$  as it is. Now to take account for  $\sigma_\psi$  in the Kamiltonian we use relation (9) and replace

$$\begin{aligned} & \int \cdots \int \Phi_\varepsilon \left( \frac{d}{dt} |\psi|^2 - \left\{ \frac{H}{0 \text{ class}}, |\psi|^2 \right\}_{\mathbf{r}, \mathbf{p}} \right) + \\ & \sum_{k=1}^n \vec{\nabla}_k \cdot \left( |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k \right) d^3 r_z \dots d^3 r_s dt = 0 \end{aligned} \quad (19)$$

Since  $G$  appearing in the Kamiltonian is any function of the generalised coordinate and momenta we can match equation (18) and (17), as a result the terms of second order in the Kamiltonian disappear and we are left with

$$\begin{aligned} \mu^{-1} K_{class} := \nu \left( \frac{H}{0 \text{ class}} + \int \cdots \int \Phi_\varepsilon \left( \frac{d}{dt} |\psi|^2 - \left\{ \frac{H}{0 \text{ class}}, |\psi|^2 \right\}_{\mathbf{r}, \mathbf{p}} \right) + \right. \\ \left. \sum_{k=1}^n \vec{\nabla}_k \cdot \left( |\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k \right) \right) d^3 r_z \dots d^3 r_s dt \end{aligned} \quad (20)$$

That's it, we have finally managed to introduce the sources (and sinks) in the Kamiltonian. We must now establish the expression of the quantum equation, that mean we have to quantize the theory. However, we will not give in this paper any commutation rules.

**Proposition 2:**

*The non linear evolution equation for  $n$  particles is*

$$\begin{aligned} i\hbar \hat{\boldsymbol{\mu}} \frac{d}{dt} |\Psi(t)\rangle = \\ \frac{1}{2} \{ \hat{\boldsymbol{\nu}}, \hat{\mathbf{H}}_q(t) + \\ \frac{1}{2} \int \cdots \int \left\{ \tilde{\Phi}_\varepsilon, \hat{\mathbf{S}}_{\psi_{\mu\nu}} \right\} d^3 \hat{\mathbf{R}}_z \dots d^3 \hat{\mathbf{R}}_s dt \} |\Psi(t)\rangle \\ + \text{Boundary conditions, Cauchy's data ...} \end{aligned} \quad (21)$$

where  $\tilde{\Phi}_\varepsilon$  is a parameter.

**Proof**

Since the variables that appear in (23) are simply functions of generalised coordinates and momenta with scalar factors, there will be an exact compatibility during the transition from the classical theory to the quantum theory by usual process of quantization (see<sup>[5][6]</sup>). First, the function  $|\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2$  become in the quantization process a function of the position operators, say

$$|\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \Leftrightarrow \hat{\mathbf{F}}_\psi \left( \hat{\mathbf{R}}_1, \dots, \hat{\mathbf{R}}_n, t \right) \quad (22)$$

Then the process of quantization of  $\sum_{k=1}^n \vec{\nabla}_k \cdot (|\psi(\mathbf{r}_1, \dots, \mathbf{r}_n, t)|^2 \vec{V}_k)$  will be univocal, for that we will have to match  $\hat{\mathbf{P}}_k$  and  $i\hbar \vec{\nabla}_k$ . Let's write  $\hat{\mathbf{J}}_\psi(t)$  the operator resulting from that quantification process. So with the help of the principle of correspondance and symetrization postulate we transform finally the sources (and sinks) as follows

$$\begin{aligned} & \left( \frac{d}{dt} |\psi|^2 - \left\{ H_{0 \text{ class}}, |\psi|^2 \right\} + \vec{\nabla}_\psi \cdot \vec{J}_\psi \right) \Leftrightarrow \\ & \frac{1}{2} \left\{ \frac{d}{dt}, \hat{\mathbf{F}}_\psi \left( \hat{\mathbf{R}}_1, \dots, \hat{\mathbf{R}}_n, t \right) \right\}_{\mathbf{r}, \mathbf{p}} + \\ & \frac{i}{\hbar} \left[ \hat{\mathbf{H}}_q(t), \hat{\mathbf{F}}_\psi \left( \hat{\mathbf{R}}_1, \dots, \hat{\mathbf{R}}_n, t \right) \right] + \hat{\mathbf{J}}_\psi(t) = \hat{\mathbf{S}}_{\psi \mu\nu} \end{aligned} \quad (23)$$

where we have matched  $H_{0 \text{ class}}$  and  $\hat{\mathbf{H}}_q(t)$  to preserve the isomorphism between the Hilbert space of kets and the  $L^2$  space. Henceforward brackets[,] and {, } in the expressions stand for commutator and anticommutator. Since we can choose the well-ordered function  $\tilde{\Phi}_\varepsilon$  we can always construct one in such a way that finally the process of quantization will be univocal. Write this unique corresponding operator  $\tilde{\Phi}_\varepsilon \left( \hat{\mathbf{R}}_1, \dots, \hat{\mathbf{R}}_k, \hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_n, \varepsilon, G, \hbar, c, k_b, t \right)$ , the product  $\tilde{\Phi}_\varepsilon \times$  "sources and sinks" in the Kamiltonian must be symetrized, so we will have

$$\Phi_\varepsilon \left( \frac{d}{dt} |\psi|^2 - \left\{ \begin{matrix} H \\ 0 \end{matrix} \right\}_{class}, |\psi|^2 \right)_{\mathbf{r}, \mathbf{p}} + \vec{\nabla}_\psi \cdot \vec{J}_\psi \right) \Leftrightarrow \frac{1}{2} \left\{ \tilde{\Phi}_\varepsilon, \hat{\mathbf{S}}_{\psi_{\mu\nu}} \right\} \quad (24)$$

Let's call  $\hat{\boldsymbol{\mu}}$  and  $\hat{\boldsymbol{\nu}}$  the matrices corresponding to the parameters  $\mu_\alpha^{-1}$  and  $\nu_\alpha$ , symmetrizing again we find the quantum Kamiltonian

$$\left\{ \begin{matrix} \hat{\boldsymbol{\mu}}, \hat{\mathbf{K}}_q \\ \hat{\boldsymbol{\nu}}, \hat{\mathbf{H}}_q(t) + \frac{1}{2} \int \dots \int \left\{ \tilde{\Phi}_\varepsilon, \hat{\mathbf{S}}_{\psi_{\mu\nu}} \right\} d^3 \hat{\mathbf{R}}_z \dots d^3 \hat{\mathbf{R}}_s dt \end{matrix} \right\} \quad (25)$$

The  $\hat{\boldsymbol{\mu}}$  and  $\hat{\boldsymbol{\nu}}$  operators appearing above are just matrices related to the scale factors  $\mu^{-1}$  and  $\nu$ . Since the elements of those matrices can be complex numbers we are consequently allowed to compose any kind of isometries (which can be connected with Killing vectors for Riemannian backgrounds). Finally, asking again for a principle of correspondence, we match  $K_{class} \Leftrightarrow i\hbar \frac{d}{dt}$  and let this act on the ket  $|\Psi(t)\rangle$ . We can remark that the  $\hat{\boldsymbol{\mu}}$  matrix will automatically commute with the diagonal matrix  $\tilde{I}_d (i\hbar \frac{d}{dt})$  where  $\tilde{I}_d$  is the identity.

#### 4) Conclusion & Acknowledgments

Let's summarize what we have done in this paper. Our first step was to show that there's a quantity that is a constant of the movement. From that we have found that there's a global conservation law for the system which represent the sources (and sinks) of the field, the second step was to introduce them in the kamiltonian, then we have proceeded to the quantization of the theory. The hidden variables that have emerged are obviously non commutative and the interaction is non local. The first question is: are those hidden variables the good ones? To answer that question study about the relation between our theory and Bell's theorem<sup>[7]</sup> has to be done. Since we are left with a non linear equation with a small nonlinearity we must seek for solitonic solutions of equation (21) and see if the masses found are those of real elementary particles in nature. Such solutions, if found to be in good accordance with accelerator's data, may be an evidence for that the old problem of duality wave/particle has no more meaning. The parameter  $\tilde{\Phi}_\varepsilon$  can be constructed with the help of the Vashy-Buckingham theorem. We are aware that the canonical transformations we have used will lead to deterministic motion

without presence of a brownian noise around the trajectories. If we wish to take account of stochastic formulation in our theory we should use another type of small changes in the coordinates. Now, what about gravitation ? we are of course not allowed to say that this is a step toward quantum gravity unless we prove that the graviton field is a possible solution of equation (21). If so, Einstein equations (probably a modified version of them) could then be derived from (21) (see <sup>[8]</sup>). But first further study should be done on the relativistic invariance, anyway, if there is any violation of Lorentz invariance this can't exceed infinitesimal spacetime regions. However, the theory does not exhibit background independance, so it is certainly not the good theory we are searching for, we may just use it as a toy model.

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