2. For obvious reasons, we most often employ numerical methods to solve differential equations for which the analytic solution is unknown. For example, using `dsolve` on the equation

$$\frac{dy}{dx} = \frac{1}{1 + \sin y}, \quad y(0) = -\frac{\pi}{4} \tag{1}$$

in Maple results in a transcendental equation for y with no closed form solution. In these cases, it is still possible to test the rate of convergence of a numerical solutions to the true solution even though the true solution is not known. Suppose we have a stencil with one-step error $\mathcal{O}(h^{p+1})$ and we use this stencil to solve the above initial value problem on the interval $x \in [0, x_0]$ twice: once with a stepsize $2h$ and once with stepsize h. The numeric solution generated when the stepsize is $2h$ will be denoted by $y_i^{(2h)} \approx y(2hi)$ with $i = 0, 1, 2 \ldots N$ [remember, $y = y(x)$ is the true solution]. Conversely, when the stepsize is h the numeric solution is labelled as $y_i^{(h)} \approx y(hi)$ with $i = 0, 1, 2 \ldots 2N$. Here, $h = x_0/2N$. Define a quantity

$$\mathcal{E}(h, x_0) = \sqrt{\frac{1}{N+1} \sum_{i=0}^{N} \left[y_i^{(2h)} - y_{2i}^{(h)} \right]^2}.$$

We call $\mathcal{E}$ the norm between the numeric solutions with stepsize h and $2h$.

(a) If the numeric method we are using to calculate the above norm is zero-stable, we would expect $\mathcal{E}(h, x_0) = \mathcal{O}(h^q)$ in the $h \to 0$ limit. What is the value of q? (No calculation is necessary here, just state the answer.)

(b) Write a single or multiple Maple procedures to calculate $\mathcal{E}(h, x_0)$ using numeric output from the Huen and classic 4th order Runge-Kutta method as applied to the initial value problem (1). In this case, you should define N using the `round` command.

(c) Plot $\mathcal{E}(h, x_0)$ versus h on a log-log plot for $x_0 = 2, 4, 6$ and for both stencils. Does the small h behaviour match your expectation from 2a?