INSTITUTO DE CIENCIAS FÍSICAS

Gaussian Random Field for Cosmology

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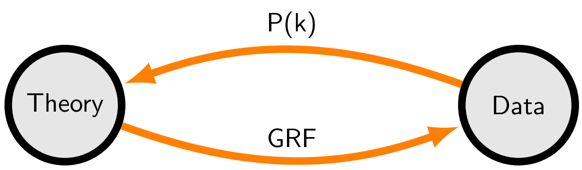
1 Introduction

The Gaussian Random Field is a powerful tool to produce random realization of random process which follow a Gaussian probability distribution function (PDF hereafter). Due to the Central Limit Theorem, random variable resulting from a sum of various different processes tend to follow a Normal distribution. We will see that generate a Gaussian Random Field is equivalent to generate random values following a Gaussian distribution (which will depend on the power spectrum) that will be the Fourier coefficients associated with the Fourier base function. We will then first introduce the Fourier Transform (i.e Fourier space decomposition). Moreover, we will see that during the early Universe (before the formation of the galaxies) the perturbations are small enough to consider independent evolution in time of the modes. For this reason, the Fourier space is of great importance for Cosmology. The Idea of this lecture is to understand how to generate a Gaussian Random Field following a given power spectrum. So, in a simple picture, the power spectrum estimation allows to link data to theoretical predictions, and the Gaussian Random Field (GRF) allows to generate data from a given power spectrum, mimicking the 2-pt statistics from a given theory.



The power spectrum $P(\vec{k})$ is the variance of the random process governing the value of $\delta_{\vec{k}}$ which is the Fourier coefficient associated to $\vec{k}$ for a given real space field $\delta(\vec{r})$:

$$\delta(\vec{r}) = \sum_{\vec{k}} \delta_{\vec{k}} e^{i\vec{k}.\vec{r}}. \tag{1}$$

In this equation, we do not specify the normalization which depends on the different existing conventions, however it do not impact the idea. We also have to keep in mind that the $\delta_{\vec{k}}$ are generally complex numbers. We will develop this expression for 1 dimension and then we will generalize to 2 and 3 dimensions.

2 Correlation Function and Power Spectum

The Probability Distribution Function of a stochastic process contains the whole information about it. If you know this function you can derive all the quantities you want in order to compare them with data. However, most of the time we do not have access to this valuable information and we need estimate it. The correlation function is a powerful tool to do it. We can evaluate the N-pt correlation function in order to access to the different moment of the underlying probability distribution. On the other hand, we can also use their corresponding Fourier transform (Power Spectrum for the 2-pt CF, Bispectrum for the 3-pt CF, Trispectrum for the 4-pt CF...)

We will start to introduce the 2pt-correlation function which is more intuitive than the power spectrum at first look but we will more focus on the later for the gaussian Random Field use.

2.1 2pt Correlation Function

A way to characterize a random process is to estimate the correlation functions (different orders will correspond to the number of points we use). The simplest and more important is the 2-pt correlation function. If we have a random distribution of points with a mean density $\bar{n} = N/V$; where N is the number of points in the total volume V; the number of points we can expect to see in a small volume dV is given by : $dN = \bar{n}dV$. If we consider two different small volume dV_1 and dV_2 we expect to observe in average $dN_1 = \bar{n}dV_1$ and $dN_2 = \bar{n}dV_2$ points respectively. So, we expect to get an average number of pairs of points between the two volumes equal to:

$$dP_{1,2} = dN_1 \times dN_2 = \bar{n}^2 dV_1 dV_2. \tag{2}$$

Because we generally use only the counting around existing points, we are interested in the number of pairs existing between this point and the points inside a small volume dV. It directly corresponds to the number of expected points in the later volume:

$$dP = 1 \times dN = \bar{n}dV. \tag{3}$$

If the distribution is not exactly random in positions, we expect to obtain a different quantity. So we let the possibility to have an excess or default respect to the randomly expected value writing:

$$dP(\vec{r}) = 1 \times dN = \bar{n} dV[1 + \xi(\vec{r})], \tag{4}$$

where $\xi(\vec{r})$ is the 2-pt correlation function. If the distribution of the points is random, the number of pairs will be compatible with $\bar{n}dV$ and so the correlation function will be null. In the other case, we will find excess and defaults in particular directions and orientations. If we consider an isotropic distribution (like the Universe if we believe in the Cosmological Principle) the deviation from the random expected number of pair have to be independent of the orientation and so will depend only on the distance $|\vec{r}|$. Moreover, in the case isotropy, we can directly consider the all shell over the point with radius r . So we can recast the equation using the volume in the shell as:

$$dP(r) = \bar{n} \times 4\pi r^2 dr[1 + \xi(r)], \tag{5}$$

which is the most common way to express the 2pt-correlation function in cosmology. In all the reasoning we done before we use the number of pair of points we expect to measure so the natural way to estimate the correlation function will be using the pair counts at each scale r and compare it with the expected value for a random distribution. In the case of a simple realization, we can estimate the correlation function as:

$$\hat{\xi}(r) = \frac{DD(r)}{N\bar{n} \times 4\pi r^2 dr} - 1, \tag{6}$$

where $DD(r)$ is the number of pairs we count at a distance $\in [r, r + dr]$ considering the N Data points. As simple case, we refer to a periodic box for which there is no limits in the pair counting. Indeed, we can alway draw a complete shell around each point in the limit of the size of this box. However, the reality is different and the expected number of pairs for the random realization is in general impossible to evaluate theoretically. For this reason, we create a random catalog reproducing the geometry containing the data points and we can then compare the pair counts between the data $DD(r)$ and the random realization $RR(r)$ as:

$$\hat{\xi}(r) = \frac{N \times DD(r)}{N \times RR(r)} - 1 = \frac{DD(r)}{RR(r)} - 1. \tag{7}$$

Moreover, we in order to reduce the variance in the estimation of $\hat{\xi}(r)$, we can increase the density of the random sample. We also need to take in to account that we will measure more pairs in the random than in the data points. If we multiply the density by a factor β then we will have $\beta N(\beta N - 1) \approx \beta^2 N^2$ pairs when we will measure $N(N - 1) \approx N^2$ pairs for the data:

$$\hat{\xi}(r) = \beta^2 \frac{DD(r)}{RR(r)} - 1, \qquad \beta \approx \frac{\bar{n}_{random}}{\bar{n}_{data}}. \tag{8}$$

It exists different estimators to evaluate the 2pt-correlation function and the one optimizing the variance and the bias of the estimation is provided by **?**:

$$\hat{\xi}_{L-S}(r) = \frac{DD(r) - 2DR(r) + RR(r)}{RR(r)}, \tag{9}$$

where $DR(r)$ is the number of pairs we can do between points from the random and the points from the data. It is possible to do it since we reproduce the geometry of the data in the random catalog. While the form looks very simple, the demonstration to show the efficiency of this estimator is pretty hard and developed in the reference.

We will not develop in more details the 2-pt correlation function since we have enough material to understand the Gaussian Random Field.

2.2 Power Spectrum

In this section we will present how to calculate a power spectrum from 3 dimension statistically isotropic data. We used the power spectrum for generating the GRF without explain how to measure/estimate it. We will also briefly see how to obtain a theoretical prediction for the power spectrum in order to have an idea why we can use it as a link between theories and data.

The power spectrum contains the information over the variance of a random process. We then understand that the power spectrum exists independently of a Gaussian process, but it contains the all information only for the later case. Knowing that the inflation scenarios still allowed by the CMB constraints predicts a Gaussian or almost Gaussian field, we understand the importance of the power spectrum for the perturbations in cosmology. Moreover, we know from the CMB observations that these fluctuations are lower than 10^{-4} up to the recombination making negligible the cross mode terms contribution. It follows that we can evolve the k-mode perturbations (and so the associated wave-plane in real space) individually during the radiation era. It is what we call the linear perturbation theory.

2.2.1 Definition and Fourier Transform

The power spectrum contains the information of the variance for each corresponding Fourier k-mode. In case we have access to various realization of a same stochastic process we can calculate the Fourier Transform of each realization and obtain the distribution of the associated coefficients. Before to go further we need to briefly define the Fourier Transform and particularly the discrete transformation which is the only one we can compute on non trivial data.

Discrete Fourier Transform The Fourier space is the one formed by the sine and cosine functions, or in terms of exponential functions using the complex notations as presented in Eq.(1). We will use the later along this note. This basis is of first importance being naturally invariant under rotation and translation. We understand that under the Cosmological Principle, the information will be optimally compress in the Fourier space.

2.2.2 Ergodicity and Cosmological Principle

As all statistical object, we need to estimate it from data. In the 1D and anisotropic cases (where we do not have redundant information in a stand alone realization) we need to work with several realizations or to use the ergodicity of the data. The ergodicity is the assumption of that various parts of the data are independent realizations of the same random process. In case of the 1D temporal case, it is equivalent to consider that the value $x(t)$ for each time t follows the same distribution allowing to infer the underlying probability distribution function (PDF) measuring the different moments over a long time serie of the data. In case of non-ergodicity we need to have access to different realizations $x_i(t)$ of the same random process and infer the different moments of the PDF at each time. In order to understand this point let us define properly the centered moments as:

$$\mu_1 = \int_{-\infty}^{\infty} x f(x) dx, \tag{10}$$

$$\mu_n = \int_{-\infty}^{\infty} (x-\mu_1)^n f(x) dx \quad \text{for n} > 1. \tag{11}$$

The way to determine them is to create an estimator over data expecting that the used sample will be representative enough of the underlying PDF $f(x)$. The estimators for the first two moments are given by:

$$\tilde{\mu}_1 = \frac{1}{N}\sum_{i=1}^{N} X_i; \qquad \tilde{\mu}_2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i-\tilde{\mu}_1)^2 \tag{12}$$

where the pre-factor for $\tilde{\mu}_2$ is obtain using Jacknife calculation and show that the creation of an estimator is highly non-trivial. However, this topic is not the subject of this note and we will not discuss it. We will focus on the data we use to make the estimation. If we do not have any idea of the underlying process which generate the data, we should use different realization sets and do the summations over the number of these realizations. So N should be the number of the realizations and we should have as much estimations of $\tilde{\mu}_1$ and $\tilde{\mu}_2$ as number of points in each realizations. In the real case, we almost never have access to independent realizations and we use the data of one realizations as independent realizations of the underlying process. This assumption is called the ergodicity. So for the 1D time series, it is equivalent to do the summation over the time using just one realization (one time series). Considering now the 2D and 3D spatial cases, evaluate the mean density of a random field is done doing the summation over the different coordinates. However, when the CP stipulate that the Universe is statistically hogeneous it implies that the mean density measured over various realizations of universe is independent of the position $\vec{r}$:

$$\bar{\rho} = \bar{\rho}(\vec{r}) = \langle \rho(\vec{r}) \rangle_\Omega, \qquad \Omega \,\text{being the ensemble of universes realizations.} \tag{13}$$

In cosmology, the ergodicity is tacitly used most of the time and one can be confused at the time to think about the Cosmological Principle. This principle postulate that "the Universe is statistically homogeneous and isotropic" while we can often listen that "the Universe is homogeneous and isotropic". The later proposition is obviously wrong if we do not add any precision about some specific scales. Our existence or the fact that there are structures in the Universe are contrary to this proposition, The term "statistically" is of great importance and stipulate that our Universe is only one realization of stochastic process and that if we have access to different universes resulting of the same process then the estimation of the mean density over the all realizations should be independent of the position.

2.2.3 Estimation of the power spectrum

We introduced all the concept and we can focus on the Power Spectrum definition, which is the Fourier transform of the 2pt correlation function. Another way to write the correlation between two random vectors $\boldsymbol{X} = \{X_0, ..., X_n\}$ and $\boldsymbol{Y} = \{Y_0, ..., Y_n\}$ is in term of the covariance:

$$C(\boldsymbol{X}, \boldsymbol{Y}) = \left\langle (X_i - \bar{X}).(Y_i - \bar{Y}) \right\rangle, \tag{14}$$

where the mean quantities vanish when considering contrast density variables. The power spectrum is the Fourier transform of this simple object and reads:

$$\begin{aligned} P(\vec{k}) &= FFT[\xi(\vec{r})] = FFT[\langle \delta(\vec{x}).\delta(\vec{x}+\vec{r})\rangle_{\vec{x}}], \\ P(\vec{k}) &= \frac{1}{(2\pi)^3}\left\langle \hat{\delta}(\vec{k}).\hat{\delta}(\vec{k}')\right\rangle_\Omega \delta^D(\vec{k}+\vec{k}'), \end{aligned} \tag{15}$$

where $\hat{\delta}(\vec{k})$ is the Fourier transform of the contrast density field and $\delta^D(\vec{k}+\vec{k}')$ is the Delta Dirac function which appears to guaranty the homogeneity. Here, we need to have access to independent realizations of the $\hat{\delta}(\vec{k})$ which can be done dividing the volume of the data in "independent" sub volumes. So we can write simplify using the fact that the Delta Dirac function is non null only if its argument is the null vector (so when $\vec{k}' = -\vec{k}$):

$$P(\vec{k}) = \frac{1}{(2\pi)^3}\left\langle \hat{\delta}(\vec{k}).\hat{\delta}(-\vec{k})\right\rangle_\Omega. \tag{16}$$

We will see in Sec 3.2 that in case of real field (i.e. not imaginary part), that $\hat{\delta}(-\vec{k}) = \hat{\delta}(\vec{k})^*$. It comes that the Power Spectrum for the vector $\vec{k}$ reads:

$$P(\vec{k}) = \frac{1}{(2\pi)^3}\left\langle \hat{\delta}(\vec{k}).\hat{\delta}(\vec{k})^*\right\rangle_\Omega = \frac{1}{(2\pi)^3}\left\langle |\hat{\delta}(\vec{k})|^2\right\rangle_\Omega \tag{17}$$

Moreover, the statistical isotropy also apply here. As we will see further, the $\hat{\delta}(\vec{k})$ are the amplitude and phases of the pane wave with frequency $|\vec{k}|$ and propagation vector $\vec{k}/|\vec{k}|$. The isotropy stipulate that the information do not depend on the orientation which implies that the information bring by each vector $\vec{k}$ such as $|\vec{k}| = k_1$ is an independent realization of the same stochastic process at the scale k_1. Then, we can reduce the isotropic Power Spectrum definition as:

$$P(k) = \frac{1}{(2\pi)^3} \left\langle |\hat{\delta}(\vec{k})|^2 \right\rangle_{|\vec{k}|=k}$$

3 Gaussian Random Field

Know we introduce properly the Fourier Transform and the power spectrum we have all the tools in order to understand and generate Gaussian Random Fields. Contrary to the estimation of the power spectrum, there is no difference on the GRF generation considering isotropic and anisotropic cases. So no distinction is needed. We first present the 1D case before to extend to the 2D and 3D cases. As we will see, there is no main differences between the different dimensions except the visualization. However, the faculty to visualize is important to infer and interpret the results for the structure formation. The 1D case will help to accustom with the random generation. We can particularly use it to support the Press and Schechter formalism in the Halo Mass Function generation. Then, we will focus on the 2D case which corresponds to the Cosmic Microwave Background space. We will first work on the wave-plane approximation before to introduce the spherical decomposition using the Legendre polynomials. It will be helpful for the $C_\ell(X_1, X_2)$ cross correlation where X_1 and X_2 are the perturbations measurements from the CMB's photons for the temperature, the E-modes or B-modes of polarizations. Finally we will treat the 3D case which is particularly in relation with the Cosmological and Astrophysical (up to galaxy scales) simulations. Indeed, the initial conditions of a large scale simulations are relied to the primordial power spectrum or with the CMB's power spectrum.

3.1 1D-Gaussian Random Field

We start with the 1D case which allows to write simply the relations we need. First, we remind that we generally use a central random field ($i.e$ the mean of this field is 0) reason why we will use the notation δ in all this lecture. For example, we use the density perturbation defined as $\delta_\rho = \frac{\rho - \bar{\rho}}{\bar{\rho}}$. The time series analysis in finance use a lot the "return" variable which corresponds to te relative variation of the price at each time step. In both cases, the mean of the variable is 0.

Anyway, in one dimension, we can rewrite Eq. 1 as

$$\delta(r) = \sum_{k=-k_{max}}^{k_{max}} \delta_k e^{ik.r}, \quad (18)$$

$$\delta(r) = \sum_{k=k_{min}}^{k_{max}} \left(\delta_k e^{ik.r} + \delta_{-k} e^{-ik.r}\right) + \delta_0 \underbrace{e^{i0.r}}_{=1} \quad (19)$$

where we regroup the terms in k and $-k$ together and where we exit the $k = 0$ term from the sum. We can see that the term $\delta_{k=0}$ contribute as a constant and so will just shift the field $\delta(r)$ or can be interpreted as the mean of the density field. If we use a contrast density field variable ,i.e $\delta = \frac{x - \bar{x}}{\bar{x}}$ where x is a stochastic variable, the mean is null by definition.

3.2 Condition for real field (no imaginary part)

In general, we want to generate a real gaussian random field in physics......because most of the natural field are reals. So we can look for the general condition which guaranty that $\delta(r)$ is real. We will start from the result and verify that it always allows the reality of $\delta(r)$. This condition is :

$$\delta_{-k} = \delta_k^*. \quad (20)$$

Indeed, restarting from Eq.18 and including the condition Eq. 20 we get

$$\delta(r) = \sum_{k=k_{min}}^{k_{max}} \left(\delta_k e^{ik.r} + \delta_k^* e^{-ik.r}\right) + \delta_0, \quad (21)$$

$$\delta(r) = \sum_{k=k_{min}}^{k_{max}} \left(\underbrace{\delta_k e^{ik.r} + \left[\delta_k e^{ik.r}\right]^*}_{=2\times Re(\delta_k e^{ik.r})}\right) + \delta_0, \quad (22)$$

where we can see that all the terms are purely reals if we fix the shift $\delta_{k=0}$ to be real.

3.2.1 Random generation of the δ_k

Now we define the condition for real field, we can generate a Gaussian Random Field which corresponds to generate the δ_k values. In order to be more explicit, we can write the complex number δ_k with its module α_k and phase ϕ_k:

$$\delta_k = \alpha_k \times e^{i\phi_k} \quad \Rightarrow \quad \delta_k^* = \alpha_k \times e^{-i\phi_k}. \quad (23)$$

We can see that if we only generate the modules α_k letting the phases $\phi_k = 0$ we get a specific realization of a real gaussian random field.

Generate α_k The most important part of the Gaussian Random Field is contained inside the α_k. Indeed, the word "Gaussian" is associated with the probability distribution function of these variables. And because the δ_k are variable of zero mean, the only information we need is the standard deviation or the variance. As we seen during the last lecture, the variance of the process is provided by the power spectrum $P(k)$, so generate a α_k corresponds to generate a random value following the pdf:

$$\alpha_k \sim \mathcal{N}\left(0, \sigma^2 = P(k)\right).$$

The word "Random" comes from this random generation of the α_k following a "Gaussian" probability distribution with variance provided by the power spectrum.

Generate ϕ_k In general, we also need to generate the phases. However, in cosmology we do not have (at least we think that there is not) information from the phase. So we also generate the phases using a uniform distribution between 0 and 2π. One time we get a phase we have to remember to satisfy the condition Eq.20 to produce a real random field.

$$\phi_k \sim \mathcal{U}\left([0, 2\pi]\right) \quad \& \quad \phi_{-k} = -\phi_k.$$

3.2.2 Get the random field $\delta(r)$

Finally, we have to calculate the inverse Fourier transformation Eq.1 to get the real space gaussian random field. We will show basic results for various cases in order to well understand in details the different impacts we describe above.

3.2.3 Individual mode impact

In order to understand the real space gaussian random field let see the impact of the individual modes in a 1D case. Let start with a simple example in which we use a power spectrum in $P(k) = k^{-2}$. We generate the values of the δ_k, respecting the reality condition, that we show the result for various k modes in the figure 3. Because we give more importance to the lower k values, we conserve a general form at large scale. The details at small scales ($i.e$ high k values) are imprinted continuously but with diminishing amplitudes. The figure shows the individual contribution of each mode associated with the sum of all the contributions from the lower modes (i.e. $\sum_{k'\leq k} \delta_{k'} e^{ik'x}$).

3.3 2D and 3D Gaussian Random Field

One time we understand how works the generation in 1D, the generalization to 2D and 3D is relatively straightforward. We will have to specify the isotropic/anisotropic conditions and their implications on the δ_k generation. In case of isotropy, we will see the reason why we can define a power spectrum $P(k)$ which depends only on the module of $\vec{k}$ and so understand why we generally we use it in cosmology. We will mostly illustrate our purpose in 2D for graphic convenience but the results are totally similar in 3D.

3.3.1 Reality of the Gaussian Random Field in 2D and 3D

We have seen in section 3.2 that the condition for reality is $\delta_{-k} = \delta_k^*$ for 1D scalar k. We will demonstrate now that the general condition of reality is $\delta_{-\vec{k}} = \delta_{\vec{k}}^*$ independently of the dimension. So we will do the demonstration only for 3D which embed the 2D case.

$$\begin{aligned}
\delta(\vec{r}) &= \sum_{k_x=-k_{x,max}}^{k_{x,max}} \sum_{k_y=-k_{y,max}}^{k_{y,max}} \sum_{k_z=k_{z,min}}^{k_{z,max}} \left(\delta_{\vec{k}=(k_x,k_y,k_z)} e^{i(k_x.r_x+k_y.r_y+k_z.r_z)} + \delta_{-\vec{k}=(-k_x,-k_y,-k_z)} e^{-i(k_x.r_x+k_y.r_y+k_z.r_z)}\right), \\
\delta(\vec{r}) &= \sum_{k_x=-k_{x,max}}^{k_{x,max}} \sum_{k_y=-k_{y,max}}^{k_{y,max}} \sum_{k_z=k_{z,min}}^{k_{z,max}} \left(\delta_{\vec{k}=(k_x,k_y,k_z)} e^{i\vec{k}.\vec{r}} + \delta_{-\vec{k}=(-k_x,-k_y,-k_z)} e^{-i\vec{k}.\vec{r}}\right), \\
\delta(\vec{r}) &= \sum_{k_x=-k_{x,max}}^{k_{x,max}} \sum_{k_y=-k_{y,max}}^{k_{y,max}} \sum_{k_z=k_{z,min}}^{k_{z,max}} \left(\delta_{\vec{k}=(k_x,k_y,k_z)} e^{i\vec{k}.\vec{r}} + \left[\delta_{\vec{k}=(k_x,k_y,k_z)} e^{i\vec{k}.\vec{r}}\right]^*\right), \qquad (24)\\
\delta(\vec{r}) &= \sum_{k_x=-k_{x,max}}^{k_{x,max}} \sum_{k_y=-k_{y,max}}^{k_{y,max}} \sum_{k_z=k_{z,min}}^{k_{z,max}} \left(\underbrace{\delta_{\vec{k}} e^{i\vec{k}.\vec{r}} + \left[\delta_{\vec{k}} e^{i\vec{k}.\vec{r}}\right]^*}_{=2\times Re(\delta_{\vec{k}} e^{i\vec{k}.\vec{r}})}\right).
\end{aligned}$$

So we get the same result than for the 1D case and the general condition for real Gaussian random field reads:

$$\text{Reality} \quad \Longleftrightarrow \quad \delta_{-\vec{k}} = \delta_{\vec{k}}^*$$

3.3.2 Isotropic field

Moreover the Cosmological Principle postulate that the Universe is statistically Isotropic and Homogeneous. We can think quickly about the signification using the basic definition Eq.1 and see that if the information depends on the orientation of $\vec{k}$ corresponds to have

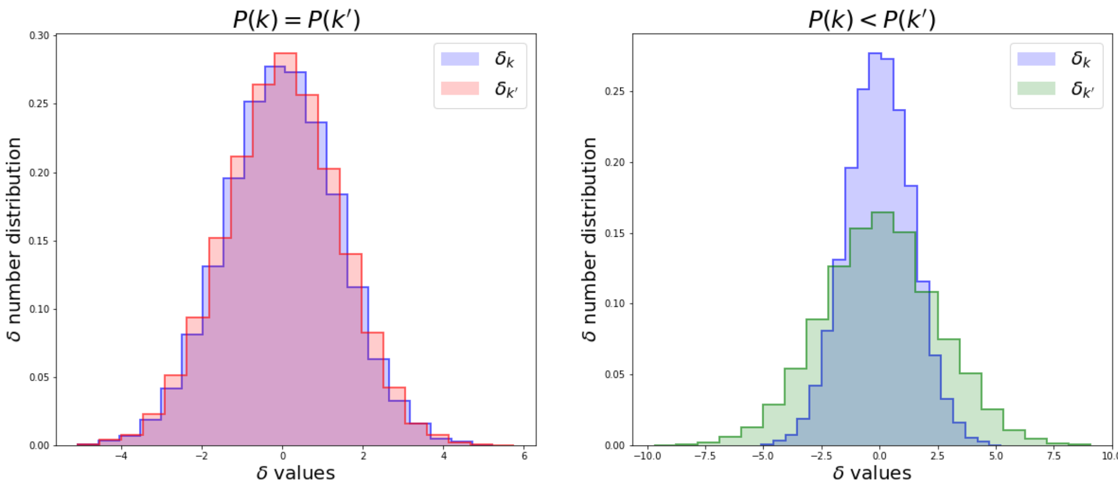


Figure 1: *Left panel* : distribution of 10000 realizations for two δ_k using the same power spectrum value. *Right panel* : distribution of 10000 realizations for two δ_k using different power spectrum values. We can see that a larger power spectrum value allows larger absolute values for the δ.

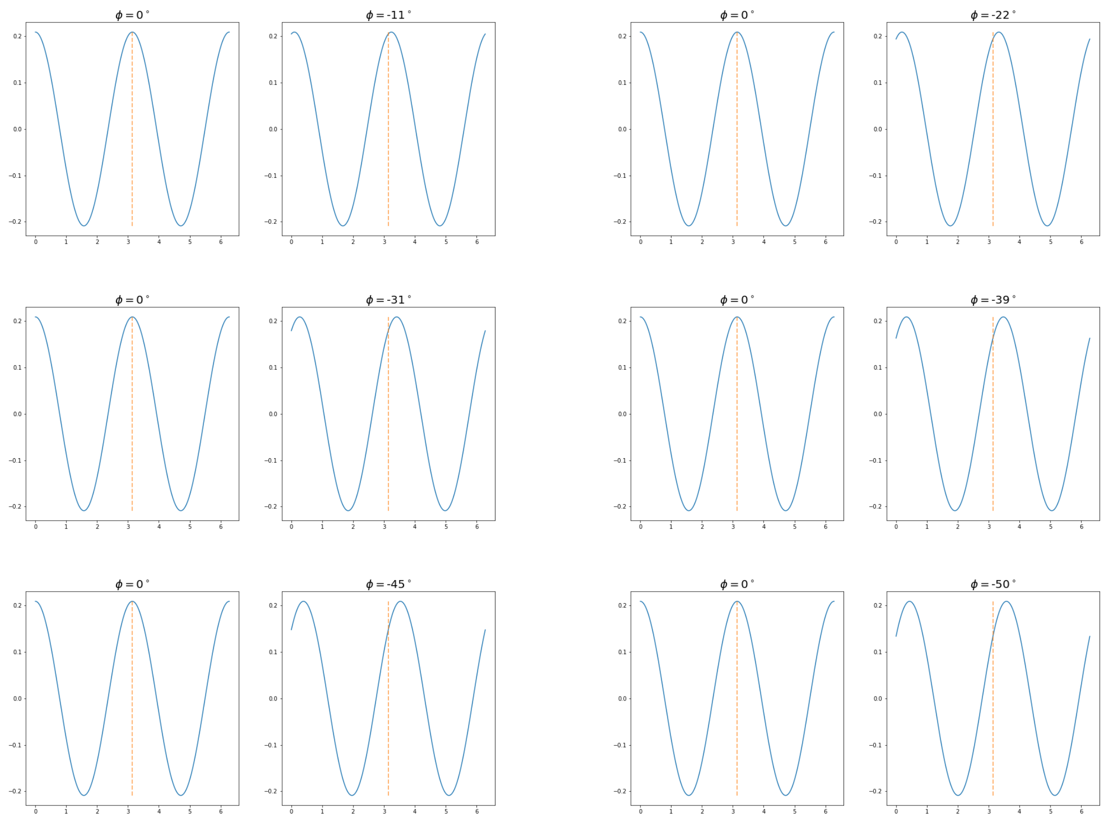


Figure 2: Impact of the phase ϕ on a given mode in real space. We can see that it lies a shift on the cosine wave.

Figure 3: Impact of the individual modes on a 1D Gaussian random field following a power spectrum in k^{-2}. Each one of the 10 plots shows the contribution of a specific mode (the right part) and the sum of all the contributions of the modes lower and equal to this mode.

a probability distributions of the $\delta_{\vec{k}}$ depending on the orientation of $\vec{k}$. As we seen in section 3.2.1, it corresponds to give in average more amplitude to the modes oriented in direction of the $\vec{k}$ vectors with greater values in the power spectrum $P(\vec{k})$. We will express it with more details. We consider two vectors $\vec{k} = (k_x, k_y)$ and $\vec{k'} = (k'_x, k'_y)$ with same module $|\vec{k}| = |\vec{k'}|$ and we will compare the impact on the real space.

If we have the same variance for the two modes (*i.e* $P(\vec{k}) = P(\vec{k'})$), then we have that the two probability distributions are equals:

$$\delta_{\vec{k}}, \delta_{\vec{k'}} \sim \mathcal{N}\left(0, \sigma^2 = P(\vec{k}) = P(\vec{k'})\right), \tag{25}$$

and so we expect to have over a large enough sample a similar distribution for two variables (Left panel in figure 1). In the figure 4, we can see the contribution in the real space field of various vectors $\vec{k}$ with same module. As we can expect considering the dot product $\vec{k}.\vec{r}$, the plane wave is oriented along $\vec{k}$. So, if we give more importance to the mode $\vec{k'}$ than to the mode $\vec{k}$ (*i.e* $P(\vec{k}) < P(\vec{k'})$), then the amplitude of the wave oriented along $\vec{k'}$ will be in general more important than the one oriented along $\vec{k}$ and generate an anisotropy in the real space. So, it appears that the condition to produce an isotropic Gaussian Random Field we need the condition:

$$\text{Isotropy} \quad \Longleftrightarrow \quad P(\vec{k}) = P(|\vec{k}|)$$

3.3.3 Impact of individual modes in 2D and 3D

The 2D Fourier decomposition corresponds to a sum of plane wave with constant values perpendicular to the vector $\vec{k}$ as we ca see an example in figure 5. The reason is simple, along an axe perpendicular to $\vec{k}$, the dot product $\vec{k}.\vec{r}$ is constant, so the term in $e^{i\vec{k}.\vec{r}}$ is constant to. In 3D, the constant value will be for all $\vec{r}$ reaching the plane perpendicular to $\vec{k}$ as shown in figure 6.

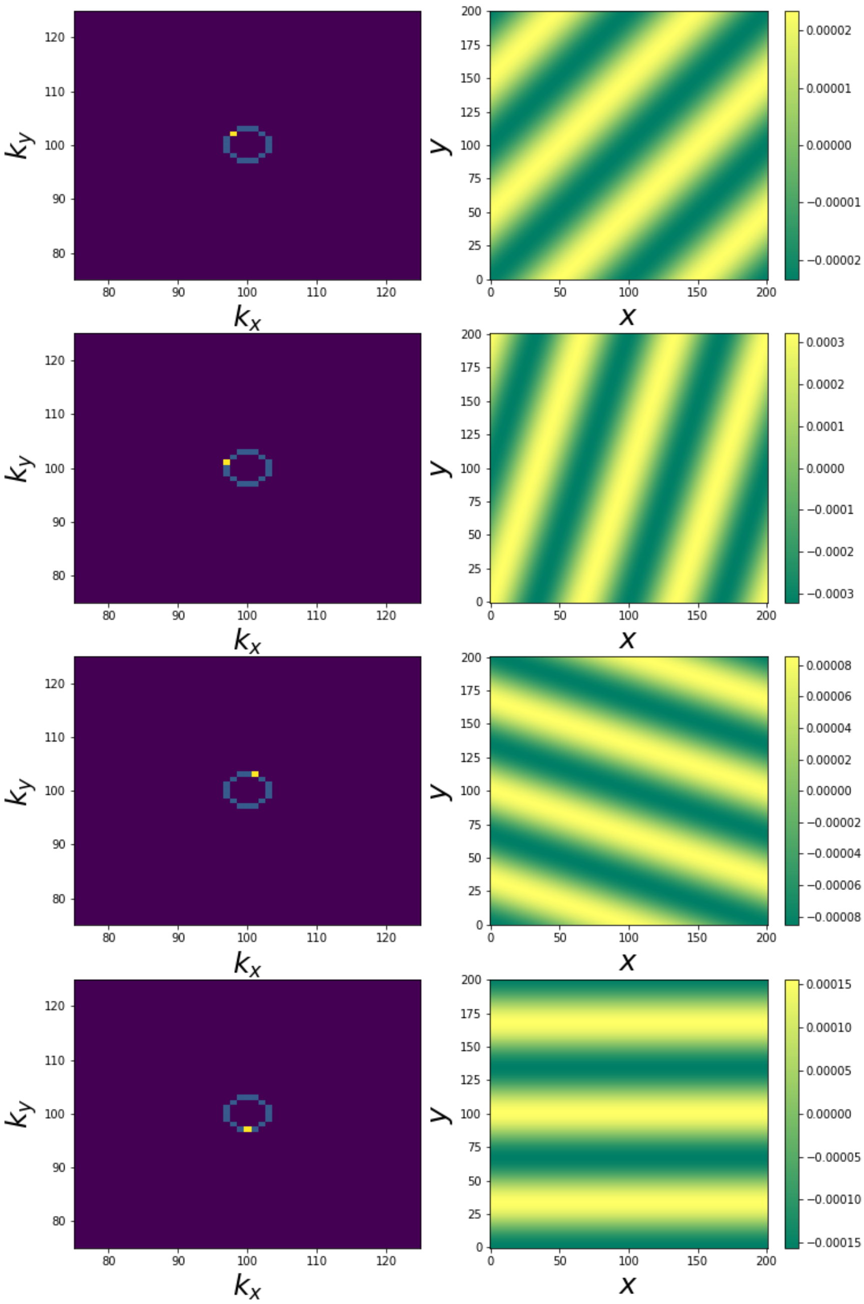


Figure 4: Effect in real space of individual pixels with same module. Same module $|\vec{k}|$ implies that the frequency/wavelength is the same for all the wave plane in the plots. The left panels show the pixel in the Fourier space which was used to generate the wave plane in the right panels. We can see that the iso-values in the right panels are perpendicular to the Fourier vector $\vec{k}$ as explained in the text. The amplitude of the wave plane fluctuations are different because the $\delta_{\vec{k}}$ differs following a random trial.

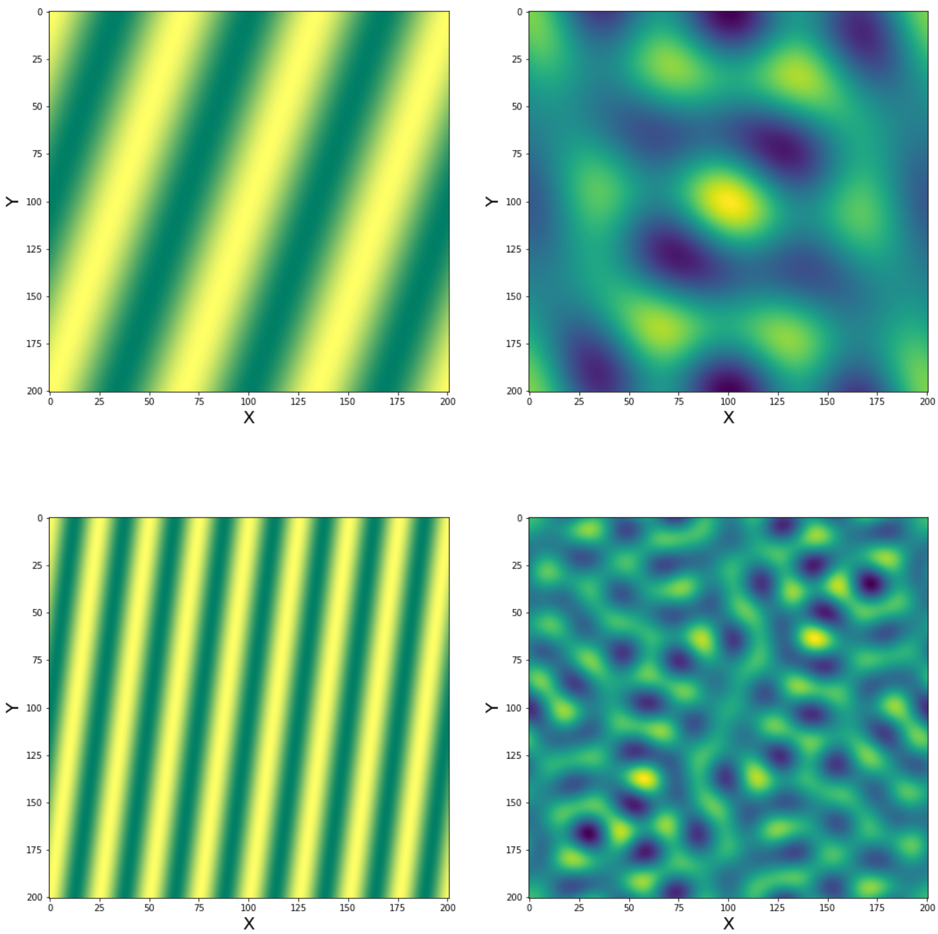


Figure 5: *Top Left panel*: Real space result for 1 pixel in Fourier space. *Top Right*: Real space result of the sum of the pixel in Fourier space with same module. The pixels used are all points which compose the circle we can see on the left panels of the figure 4. *Bottom Left and Right:*We redo the same exercise for another frequency, so another module of $\vec{k}$ which is gratter than the previous one.

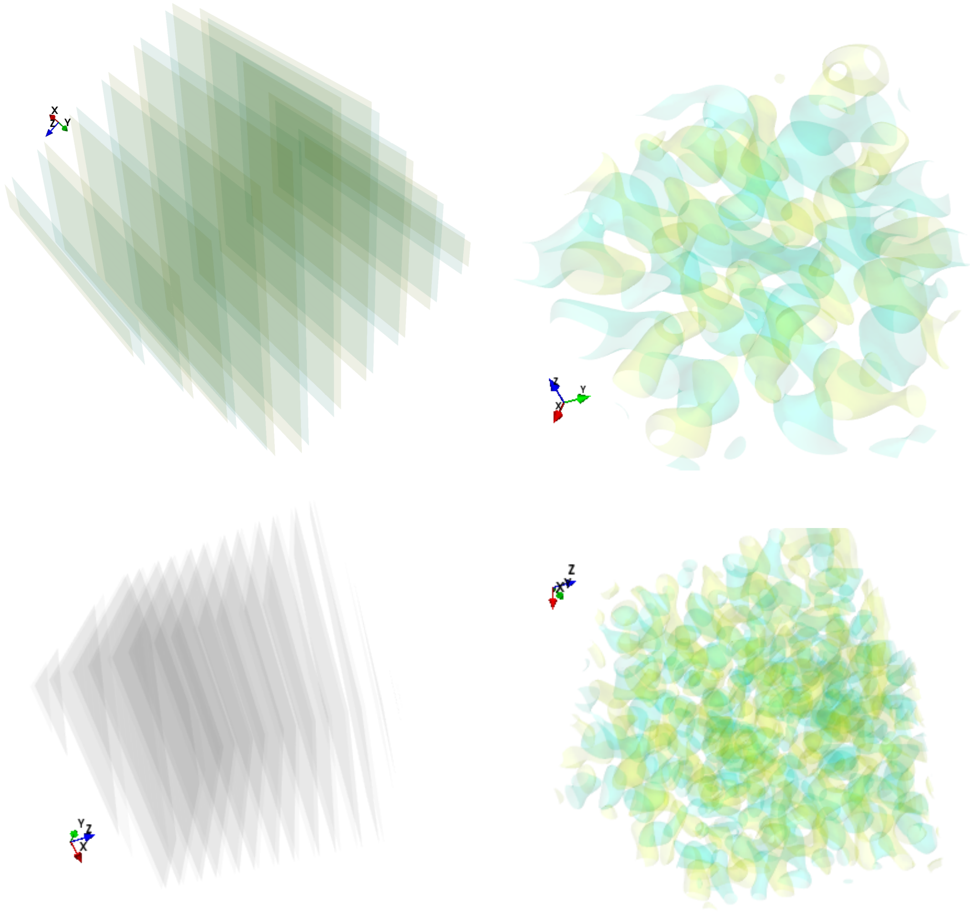


Figure 6: *Top Left panel* : Real space result for 1 pixel $\vec{k}$ in 3D. The result shows the iso-contours which corresponds to perpendicular planes as explained in the text. Each plane corresponds to the same value so the space between 2 planes correspond to the wavelength of the mode $\vec{k}$. *Top Right panel* : Real space result of the sum of all the pixels on a sphere (a shell) so the contribution of all the $\vec{k_i}$ tq $|\vec{k_i}| = k\,\forall i$. *Bottom Left and Right panels* : The same than above but for a greater module $|\vec{k}|$