Big-bounce cosmology with spinor-torsion coupling

Nikodem J. Popławski
Department of Physics, Indiana University, Swain Hall West,
*727 East Third Street, Bloomington, Indiana 47405, USA**
(Dated: November 22, 2011)

The Einstein-Cartan-Sciama-Kibble theory of gravity removes the constraint of general relativity that the affine connection be symmetric by regarding its antisymmetric part, the torsion tensor, as a dynamical variable. The minimal coupling between the torsion tensor and Dirac spinors leads to gravitational repulsion in fermionic matter at extremely high densities even without approximating it as a spin fluid. We show that such a repulsion replaces the unphysical big-bang singularity with a nonsingular big bounce that follows a contracting phase of the Universe. This scenario also naturally explains why the Universe today appears spatially flat, homogeneous and isotropic.

PACS numbers: 04.50.Kd, 11.10.Ef, 98.80.Bp, 98.80.Cq
Keywords: Einstein-Cartan-Sciama-Kibble gravity, Dirac Lagrangian, torsion, spinors, minimal coupling, nonsingular universe, big bounce, flatness problem, horizon problem.

The Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity, like general relativity, is based on the Lagrangian density of the gravitational field that is proportional to the curvature scalar R [1]. It removes, however, the GR constraint that the affine connection $\Gamma^{\,\,k}_{i\,j}$ be symmetric in its lower indices by regarding the antisymmetric part of the connection, the torsion tensor $S^k_{ij} = \Gamma^{\,\,k}_{[i\,j]}$, as a dynamical variable [2]. Varying the total Lagrangian density $-\frac{R\sqrt{-g}}{2\kappa} + \mathfrak{L}_\mathrm{m}$, where $\mathfrak{L}_\mathrm{m}$ is the Lagrangian density of matter, with respect to the contortion tensor $C_{ijk} = S_{ijk} + S_{jki} + S_{kji}$ gives the Cartan equations

$$S^j_{ik} - S_i\delta^j_k + S_k\delta^j_i = -\frac{\kappa}{2}s_{ik}^{j}, \tag{1}$$

where $S_i = S^k_{ik}$ and $s^{ijk} = \frac{2}{\sqrt{-g}}\frac{\delta\mathfrak{L}_\mathrm{m}}{\delta C_{ijk}}$ is the spin tensor. Varying the total Lagrangian density with respect to the metric tensor g_{ik} gives the Einstein equations with terms on the curvature side that are quadratic in the torsion tensor. Substituting (1) into these equations leads to the Einstein-Cartan equations $G_{ik} = \kappa(T_{ik} + U_{ik})$, where G_{ik} is the Einstein tensor, $T_{ik} = \frac{2}{\sqrt{-g}}\frac{\delta\mathfrak{L}_\mathrm{m}}{\delta g^{ik}}$ is the energy-momentum tensor, and

$$U^{ik} = \kappa\biggl(-s^{ij}_{[l}s^{kl}_{j]} - \frac{1}{2}s^{ijl}s^k_{jl} + \frac{1}{4}s^{jli}s_{jl}^{k} + \frac{1}{8}g^{ik}(-4s^l_{j[m}s^{jm}_{l]} + s^{jlm}s_{jlm})\biggr) \tag{2}$$

is the correction to the energy-momentum tensor generated by torsion and quadratic in the spin density [3]. The spin tensor also appears in T_{ik} because $\mathfrak{L}_\mathrm{m}$ depends on torsion.

In the ECSK gravity, the Dirac Lagrangian density for a free spinor ψ with mass m and minimally coupled to the gravitational field is given by $\mathfrak{L}_\mathrm{m} = \frac{i\sqrt{-g}}{2}(\bar{\psi}\gamma^i\psi_{;i} - \bar{\psi}_{;i}\gamma^i\psi) - m\sqrt{-g}\bar{\psi}\psi$, where g is the determinant of the metric tensor, γ^i are the Dirac matrices obeying $\gamma^{(i}\gamma^{k)} = g^{ik}I$, and semicolon denotes a full covariant derivative with respect to the affine connection [2, 3]:

$$\psi_{;k} = \psi_{:k} + \frac{1}{4}C_{ijk}\gamma^{[i}\gamma^{j]}\psi, \quad \bar{\psi}_{;k} = \bar{\psi}_{:k} - \frac{1}{4}C_{ijk}\bar{\psi}\gamma^{[i}\gamma^{j]}. \tag{3}$$

Colon denotes a Riemannian covariant derivative with respect to the Christoffel symbols and we use the units in which $\hbar = c = 1$, so $\kappa = 8\pi G = m_\mathrm{Pl}^{-2}$, where m_Pl is the reduced Planck mass. For a Dirac field, the spin tensor is totally antisymmetric:

$$s^{ijk} = -e^{ijkl}s_l, \quad s^i = \frac{1}{2}\bar{\psi}\gamma^i\gamma^5\psi, \tag{4}$$

where $e^{ijkl} = \frac{\epsilon^{ijkl}}{\sqrt{-g}}$, ϵ^{ijkl} is the Levi-Civita permutation symbol, and s^i is the Dirac spin pseudovector. The Cartan equations for such a field give therefore the totally antisymmetric contortion tensor [2]:

$$C_{ijk} = S_{ijk} = \frac{\kappa}{2}e_{ijkl}s^l. \tag{5}$$

---

*Electronic address: nipoplaw@indiana.edu

The torsion tensor in this case is dual to a pseudovector that is proportional to the Dirac spin pseudovector. Substituting (4) into (2) gives

$$U^{ik} = \frac{\kappa}{4}(2s^i s^k + s^l s_l g^{ik}). \tag{6}$$

Varying $\mathfrak{L}_{\rm m}$ with respect to the spinor adjoint conjugate $\bar{\psi}$ gives the Dirac equation $i\gamma^k\psi_{;k} = m\psi$ [3, 4]. The energy-momentum tensor corresponding to the Dirac Lagrangian density is, using this equation,

$$T_{ik} = \frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{;j} - \bar{\psi}_{;j}\delta^j_{(i}\gamma_{k)}\psi) - \frac{i}{2}(\bar{\psi}\gamma^j\psi_{;j} - \bar{\psi}_{;j}\gamma^j\psi)g_{ik} + m\bar{\psi}\psi g_{ik} = \frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{;j} - \bar{\psi}_{;j}\delta^j_{(i}\gamma_{k)}\psi). \tag{7}$$

Substituting (3) and (5) into (7) gives

$$T_{ik} = \frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{:j} - \bar{\psi}_{:j}\delta^j_{(i}\gamma_{k)}\psi) + \frac{\kappa}{2}(-s_i s_k + s^l s_l g_{ik}). \tag{8}$$

The combined energy-momentum tensor for a Dirac field on the right-hand side of the Einstein-Cartan equations is thus

$$T_{ik} + U_{ik} = \frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{:j} - \bar{\psi}_{:j}\delta^j_{(i}\gamma_{k)}\psi) + \frac{3\kappa}{4}s^l s_l g_{ik}, \tag{9}$$

which has been found by Kerlick [5]. This expression agrees with [6], where we derived it from the Hehl-Datta equation, which is the Dirac equation upon substituting (3), (4) and (5) [4]. The first term on the right of (9) is the GR part of the energy-momentum tensor for a Dirac field and can be macroscopically averaged at cosmological scales as a perfect fluid with the energy density ϵ and pressure p. In the comoving frame of reference, the second term on the right of (9) is equal to $-\frac{3\kappa}{4}\mathbf{s}^2 g_{ik}$, where $\mathbf{s}$ is the spatial spin psedovector which measures the spatial density of spin. The average value of its square is $\langle\mathbf{s}^2\rangle = \frac{3}{4}n^2$, where n is the fermion number density. The averaged second term on the right of (9) acts thus like a negative vacuum energy with $\tilde{\epsilon} = -\tilde{p} = -\alpha n^2$, where $\alpha = \frac{9\kappa}{16}$.

Hehl, von der Heyde, and Kerlick have used the spin-fluid approximation of matter, $s_{ijk} = s_{ij}u_k$ and $s_{ij}u^j = 0$, where u^i is the four-velocity, to show that the correction to the energy-momentum tensor due to torsion behaves like a stiff matter with a negative energy density, $\tilde{\epsilon} = \tilde{p} = -\frac{\kappa}{4}s^2$, where $s^2 = \frac{1}{2}s_{ik}s^{ik} = \frac{1}{8}n^2$ [7]. This behavior, which is valid even without spin polarization, is significant in fermionic matter at extremely high densities, where it leads to gravitational repulsion and avoidance of singularities. Kuchowicz has shown that such a repulsion replaces the big-bang singularity with a nonsingular big bounce, before which the Universe was contracting [8]. Kopczyński and Trautman have found a similar result for fermionic matter with polarized spin [9]. The spin-fluid model can be derived as the particle approximation of multiple expansion of the integrated conservation laws in the ECSK gravity [10]. The particle approximation for Dirac fields, however, is not self-consistent [11]. The spin-fluid description also violates the cosmological principle [12]. In this paper, we use the Dirac form of the spin tensor, $s_{ijk} = s_{[ijk]}$ [2, 5], which follows directly from the Dirac Lagrangian and is consistent with the cosmological principle [13]. We consider the dynamics of the early Universe, as we did for the spin-fluid form of the spin tensor in [14], and show that the minimal coupling between the torsion tensor and fermions with the Dirac form of the spin tensor also leads to gravitational repulsion and avoidance of singularities.

As in [14], we consider a closed, homogeneous and isotropic universe, described by the Friedman-Lemaître-Robertson-Walker (FLRW) metric. In the isotropic spherical coordinates, this metric is given by $ds^2 = c^2dt^2 - \frac{a^2(t)}{(1+kr^2/4)^2}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$, where $a(t)$ is the scale factor and $k = 1$ [1]. The corresponding Einstein field equations in the comoving frame for the combined energy-momentum tensor (9) become the Friedman equations (we do not write the cosmological constant which is negligible in the early Universe):

$$\dot{a}^2 + k = \frac{1}{3}\kappa\Big(\epsilon - \alpha n^2\Big)a^2, \tag{10}$$

$$\dot{a}^2 + 2a\ddot{a} + k = -\kappa\Big(p + \alpha n^2\Big)a^2, \tag{11}$$

where dot denotes the differentiation with respect to t. These equations yield the conservation law

$$\frac{d}{dt}\big((\epsilon - \alpha n^2)a^3\big) + (p + \alpha n^2)\frac{d}{dt}(a^3) = 0, \tag{12}$$

which gives

$$a^3 d\epsilon - 2\alpha a^3 n dn + (\epsilon + p)d(a^3) = 0. \tag{13}$$

As in [15], we use the values of ϵ, p and n for ultrarelativistic matter in kinetic equilibrium: $\epsilon(T) = \frac{\pi^2}{30}g_\star(T)T^4$, $p(T) = \frac{\epsilon(T)}{3}$ and $n(T) = \frac{\zeta(3)}{\pi^2}g_n(T)T^3$, where T is the temperature of the early Universe, $g_\star(T) = g_{\rm b}(T) + \frac{7}{8}g_{\rm f}(T)$, $g_n(T) = \frac{3}{4}g_{\rm f}(T)$, $g_{\rm b} = \sum_i g_i$ is summed over relativistic bosons, $g_{\rm f} = \sum_i g_i$ is summed over relativistic fermions, and g_i is the number of the spin states for each particle species i. Substituting these values to (13) gives

$$\frac{dT}{T} - \frac{3\alpha h_n^2}{2h_\star}TdT + \frac{da}{a} = 0, \tag{14}$$

where $h_\star = \frac{\pi^2}{30}g_\star(T)$ and $h_n = \frac{\zeta(3)}{\pi^2}g_n(T)$ can be assumed constant in the range of T considered.[1] Integrating (14) gives

$$a = \frac{a_0 T_0}{T}\exp\Big(\frac{3\alpha h_n^2}{4h_\star}T^2\Big), \tag{15}$$

where a_0 is the scale factor at a reference temperature T_0.

The function $a(T)$ given by (15) is not monotonic. As T increases, a decreases until T reaches a critical temperature $T_{\rm cr}$ given by $\frac{da}{dT}(T_{\rm cr}) = 0$,

$$T_{\rm cr} = \Big(\frac{2h_\star}{3\alpha h_n^2}\Big)^{1/2}, \tag{16}$$

and then increases. Since an increasing function $a(T)$ is unphysical, $a_{\rm cr} = a(T_{\rm cr}) > 0$ is the smallest allowed value of the scale factor:

$$a_{\rm cr} = a_0 T_0\Big(\frac{3e\alpha h_n^2}{2h_\star}\Big)^{1/2}. \tag{17}$$

The Universe is therefore nonsingular: $a \geq a_{\rm cr}$. For $T \gg T_{\rm cr}$, (15) reduces to $a = \frac{a_0 T_0}{T}$, which is satisfied in the radiation-dominated era.

To verify that $a_{\rm cr}$ is the minimum scale factor of the Universe, we substitute (14) into (10) without the negligible term $k = 1$, obtaining

$$\dot{T}^2\Big(\frac{1}{T^2} - \frac{3\alpha h_n^2}{2h_\star}\Big)^2 = \frac{\kappa}{3}(h_\star T^2 - \alpha h_n^2 T^4). \tag{18}$$

Denoting $\beta = T^{-1}$, $\beta_{\rm cr} = T_{\rm cr}^{-1}$ and using (16) leads to

$$|\dot{\beta}| = \sqrt{\frac{\kappa h_\star}{3}}\frac{\sqrt{\beta^2 - \frac{2}{3}\beta_{\rm cr}^2}}{\beta^2 - \beta_{\rm cr}^2}, \tag{19}$$

which yields $\beta \geq \beta_{\rm cr}$, $T \leq T_{\rm cr}$, and thus $a \geq a_{\rm cr}$ using (15). Integrating (19), together with (15) written as

$$a = \frac{a_0\beta}{\beta_0}\exp\Big(\frac{\beta_{\rm cr}^2}{2\beta^2}\Big), \tag{20}$$

gives the dynamics of the early Universe. The Universe contracts ($\dot{a} < 0$) until $\beta = \beta_{\rm cr}$ and $a = a_{\rm cr}$, and then expands ($\dot{a} > 0$). The unphysical big-bang singularity appearing in general-relativistic cosmology is replaced in the ECSK gravity by a nonsingular big bounce that follows a contracting phase of the Universe. For $\beta \gg \beta_{\rm cr}$, (19) and (20) give $a \propto \beta \propto t^{1/2}$, which is characteristic to the radiation-dominated era.

One can show that the condition $\dot{a} = 0$, defining the scale factor at a locally stationary state $a = a_{\rm st}$, is never satisfied. This condition, using (10), would be satisfied at a temperature $T_{\rm st}$ given by

$$h_\star T^4 - \alpha h_n^2 T^6 - \frac{3}{\kappa a^2} = 0. \tag{21}$$

---

[1] For constant values of $g_\star$ and g_n, the relations $\epsilon \propto T^4$, $p = \frac{\epsilon}{3}$ and $n \propto T^3$ are consistent with a relation $\frac{dn}{n} = \frac{d\epsilon}{\epsilon+p}$ used in [14]. Interestingly, if $\tilde{\epsilon} = \tilde{p} \propto n^2$ as in [7, 14], then we also have $\frac{dn}{n} = \frac{d(\epsilon+\tilde{\epsilon})}{\epsilon+\tilde{\epsilon}+p+\tilde{p}}$.

However, $T_{\rm st} > T_{\rm cr}$, so the Universe never reaches $T_{\rm st}$. At the minimum scale factor, the Universe undergoes a bounce from $\dot{a} = -v$ to $\dot{a} = v$, where

$$v = \left(\frac{\kappa}{3}(h_\star T_{\rm cr}^4 - \alpha h_n^2 T_{\rm cr}^6)a_{\rm cr}^2 - 1\right)^{1/2} = \left(\frac{32eh_\star^2}{243h_n^2}(a_0 T_0)^2 - 1\right)^{1/2}. \quad (22)$$

The density parameter at the bounce is given by

$$\Omega = 1 + \frac{243h_n^2}{32eh_\star^2(a_0 T_0)^2}. \quad (23)$$

The values a_0 and T_0 for the present Universe show that v is extremely large, while $\Omega - 1$ is extremely small in magnitude, on the order of the corresponding values in [14]. Therefore, the ECSK theory of gravity and Dirac spinors minimally coupled to the gravitational field naturally explain why the Universe today appears spatially flat, homogeneous and isotropic, without introducing hypothetical matter fields of cosmological inflation.

---

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).
[2] T. W. B. Kibble, J. Math. Phys. (N.Y.) **2**, 212 (1961); D. W. Sciama, in *Recent Developments in General Relativity* (Pergamon, Oxford, 1962), p. 415; D. W. Sciama, Rev. Mod. Phys. **36**, 463 (1964); D. W. Sciama, Rev. Mod. Phys. **36**, 1103(E) (1964).
[3] F. W. Hehl, Phys. Lett. A **36**, 225 (1971); F. W. Hehl, Gen. Relativ. Gravit. **4**, 333 (1973); F. W. Hehl, Gen. Relativ. Gravit. **5**, 491 (1974); F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976); E. A. Lord, *Tensors, Relativity and Cosmology* (McGraw-Hill, New Delhi, 1976); V. de Sabbata and C. Sivaram, *Spin and Torsion in Gravitation* (World Scientific, Singapore, 1994); I. L. Shapiro, Phys. Rep. **357**, 113 (2002); R. T. Hammond, Rep. Prog. Phys. **65**, 599 (2002); N. J. Popławski, arXiv:0911.0334.
[4] F. W. Hehl and B. K. Datta, J. Math. Phys. (N.Y.) **12**, 1334 (1971).
[5] G. D. Kerlick, Phys. Rev. D **12**, 3004 (1975).
[6] N. J. Popławski, Ann. Phys. (Berlin) **523**, 291 (2011).
[7] F. W. Hehl, P. von der Heyde, and G. D. Kerlick, Phys. Rev. D **10**, 1066 (1974); I. S. Nurgaliev and W. N. Ponomariev, Phys. Lett. B **130**, 378 (1983); M. Gasperini, Phys. Rev. Lett. **56**, 2873 (1986).
[8] B. Kuchowicz, Gen. Relativ. Gravit. **9**, 511 (1978).
[9] W. Kopczyński, Phys. Lett. A **39**, 219 (1972); A. Trautman, Nature (Phys. Sci.) **242**, 7 (1973).
[10] K. Nomura, T. Shirafuji, and K. Hayashi, Prog. Theor. Phys. **86**, 1239 (1991).
[11] N. J. Popławski, Phys. Lett. B **690**, 73 (2010).
[12] M. Tsamparlis, Phys. Lett. A **75**, 27 (1979).
[13] P. Minkowski, Phys. Lett. B **173**, 247 (1986).
[14] N. J. Popławski, Phys. Lett. B **694**, 181 (2010); N. J. Popławski, Phys. Lett. B **701**, 672(E) (2011).
[15] N. J. Popławski, arXiv:1105.6127.