The factor in large parentheses is equal to $E^2 - \omega^2 + i(E^2 + \omega^2)\epsilon$, and we can absorb the positive coefficient into ϵ to get $E^2 - \omega^2 + i\epsilon$.

Now it is convenient to change integration variables to

$$\widetilde{x}(E) = \widetilde{q}(E) + \frac{\widetilde{f}(E)}{E^2 - \omega^2 + i\epsilon} \, . \tag{7.7}$$

Then we get

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \left[\widetilde{x}(E)(E^2 - \omega^2 + i\epsilon)\widetilde{x}(-E) - \frac{\widetilde{f}(E)\widetilde{f}(-E)}{E^2 - \omega^2 + i\epsilon} \right] . \tag{7.8}$$

Furthermore, because eq. (7.7) is just a shift by a constant, $\mathcal{D}q = \mathcal{D}x$. Now we have

$$\begin{aligned} \langle 0|0\rangle_f &= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right] \\ &\quad \times \int \mathcal{D}x \, \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \, \widetilde{x}(E)(E^2 - \omega^2 + i\epsilon)\widetilde{x}(-E) \right] . \end{aligned} \tag{7.9}$$

Now comes the key point. The path integral on the second line of eq. (7.9) is what we get for $\langle 0|0\rangle_f$ in the case $f = 0$. On the other hand, if there is no external force, a system in its ground state will remain in its ground state, and so $\langle 0|0\rangle_{f=0} = 1$. Thus $\langle 0|0\rangle_f$ is given by the first line of eq. (7.9),

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right] . \tag{7.10}$$

We can also rewrite $\langle 0|0\rangle_f$ in terms of time-domain variables as

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} dt\, dt' \, f(t)G(t-t')f(t') \right] , \tag{7.11}$$

where

$$G(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2 + \omega^2 - i\epsilon} \, . \tag{7.12}$$

Note that $G(t-t')$ is a Green's function for the oscillator equation of motion:

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) G(t-t') = \delta(t-t') \, . \tag{7.13}$$

This can be seen directly by plugging eq. (7.12) into eq. (7.13) and then taking the $\epsilon \to 0$ limit. We can also evaluate $G(t-t')$ explicitly by treating the integral over E on the right-hand side of eq. (7.12) as a contour integral in the complex E plane, and then evaluating it via the residue theorem. The result is

$$G(t-t') = \frac{i}{2\omega}\exp\Big(-i\omega|t-t'|\Big)\ . \tag{7.14}$$

Consider now the formula from section 6 for the time-ordered product of operators. In the case of initial and final ground states, it becomes

$$\langle 0|\mathrm{T}Q(t_1)\ldots|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta f(t_1)}\ldots\langle 0|0\rangle_f\Big|_{f=0}\ . \tag{7.15}$$

Using our explicit formula, eq. (7.11), we have

$$\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle &= \frac{1}{i}\,\frac{\delta}{\delta f(t_1)}\,\frac{1}{i}\,\frac{\delta}{\delta f(t_2)}\,\langle 0|0\rangle_f\Big|_{f=0} \\
&= \frac{1}{i}\,\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty}dt'\,G(t_2-t')f(t')\right]\langle 0|0\rangle_f\Big|_{f=0} \\
&= \Big[\tfrac{1}{i}G(t_2-t_1) + (\text{term with } f\text{'s})\Big]\langle 0|0\rangle_f\Big|_{f=0} \\
&= \tfrac{1}{i}G(t_2-t_1)\ .
\end{aligned} \tag{7.16}$$

We can continue in this way to compute the ground-state expectation value of the time-ordered product of more $Q(t)$'s. If the number of $Q(t)$'s is odd, then there is always a left-over $f(t)$ in the prefactor, and so the result is zero. If the number of $Q(t)$'s is even, then we must pair up the functional derivatives in an appropriate way to get a nonzero result. Thus, for example,

$$\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\Big[&G(t_1{-}t_2)G(t_3{-}t_4) \\
+\,&G(t_1{-}t_3)G(t_2{-}t_4) \\
+\,&G(t_1{-}t_4)G(t_2{-}t_3)\Big].
\end{aligned} \tag{7.17}$$

More generally,

$$\langle 0|\mathrm{T}Q(t_1)\ldots Q(t_{2n})|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} G(t_{i_1}{-}t_{i_2})\ldots G(t_{i_{2n-1}}{-}t_{i_{2n}})\ . \tag{7.18}$$

Problems

7.1) Starting with eq. (7.12), do the contour integral to verify eq. (7.14).

7.2) Starting with eq. (7.14), verify eq. (7.13).