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Supplement 1 to the GUM

Propagation of distributions using a Monte-Carlo method

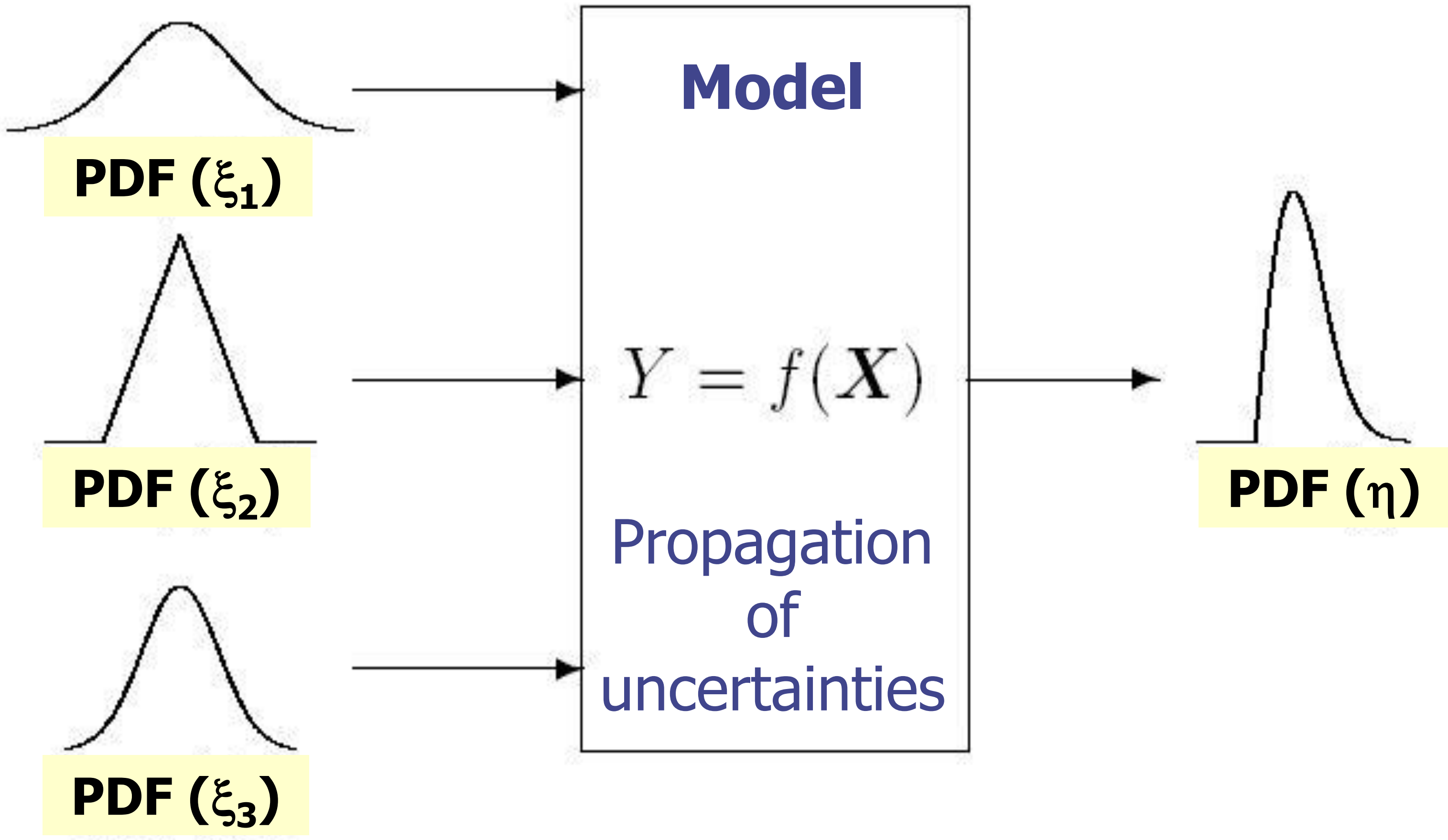
(JCGM 101:2008)

Dr M. Maeck Belgian National Metrology

Supplement 1 to the GUM

- **1. Comparison GUM – GUM+1**
- 2. Monte-Carlo methods
- 3. Evaluation of measurement uncertainty using a MC method
- 4. Calibration of a thermometer (GUM H3)

Comparison GUM – GUM+1 (similarities)



PDF: probability density function

Comparison GUM – GUM+1 (differences)

- **1. Propagation of uncertainties:**
  - GUM: series expansion
  - GUM+1: numerical method (Monte Carlo simulations)
- **2. The output PDF:**
  - Supposed to be normal or « Student like » for the GUM (υ = ? ; k = ?)
  - Obtained explicitly for the GUM+1 (neither υ nor k needed)

Comparison GUM – GUM+1 (advantages of GUM+1)

- **1. Less restrictions:**
  - The model can be strongly nonlinear
  - The output PDF can be almost anything

- **2. Easy to program:**
  - No computation of derivatives
  - The programming logic is very simple

- **3. Interpretation:**
  - Based on the CDF
  - The confidence interval can be « personalized »

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Monte-Carlo methods (principle)

Monte Carlo methods (or Monte Carlo experiments) are a class of computational algorithms that rely on repeated random sampling to compute their results. Monte Carlo methods are often used in simulating physical and mathematical systems. These methods are most suited to calculation by a computer and tend to be used when it is infeasible to compute an exact result with a deterministic algorithm. This method is also used to complement the theoretical derivations.

(http://en.wikipedia.org/wiki/Monte_Carlo_method)

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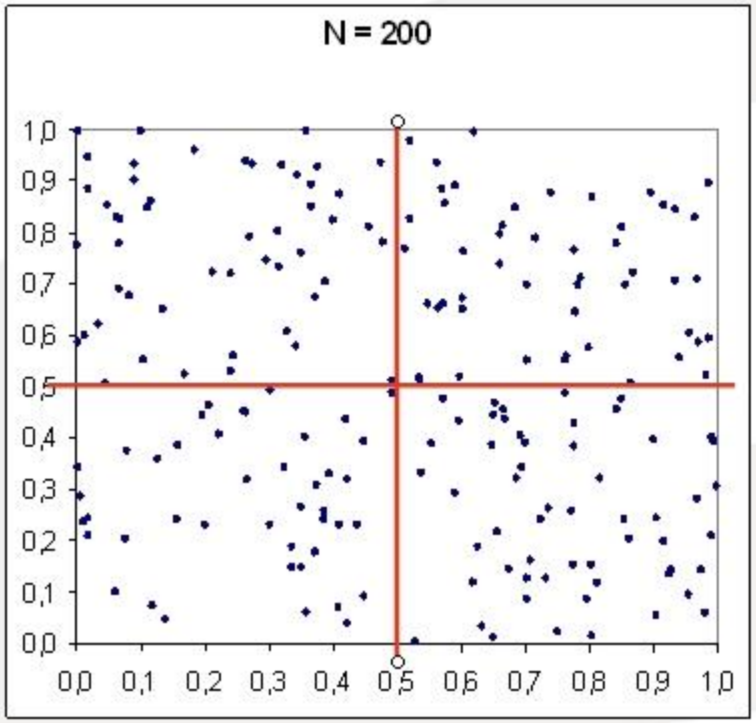
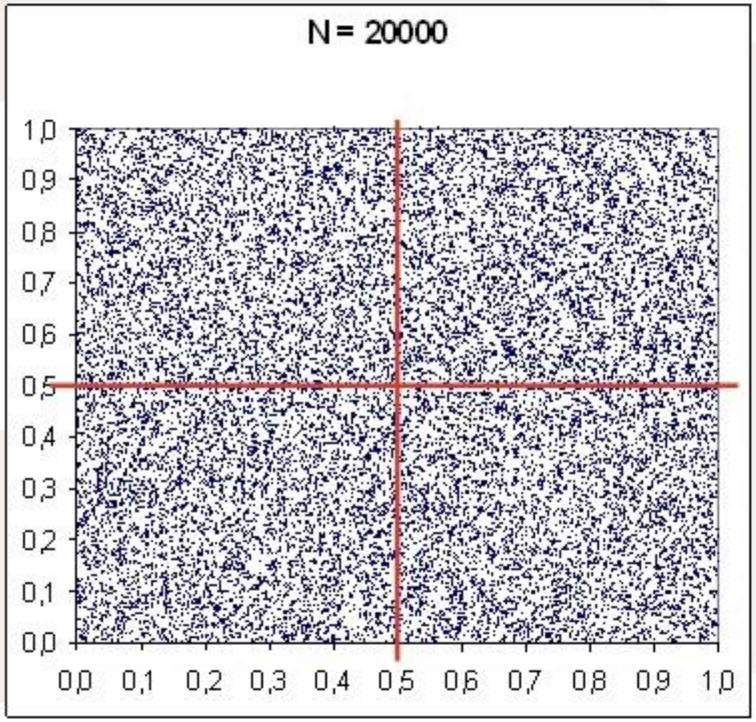
Monte-Carlo method (example)





$$\frac{N_{surface}}{N_{square}} \approx \frac{S_{surface}}{S_{square}} = \frac{P_{surface}}{P_{square}}$$

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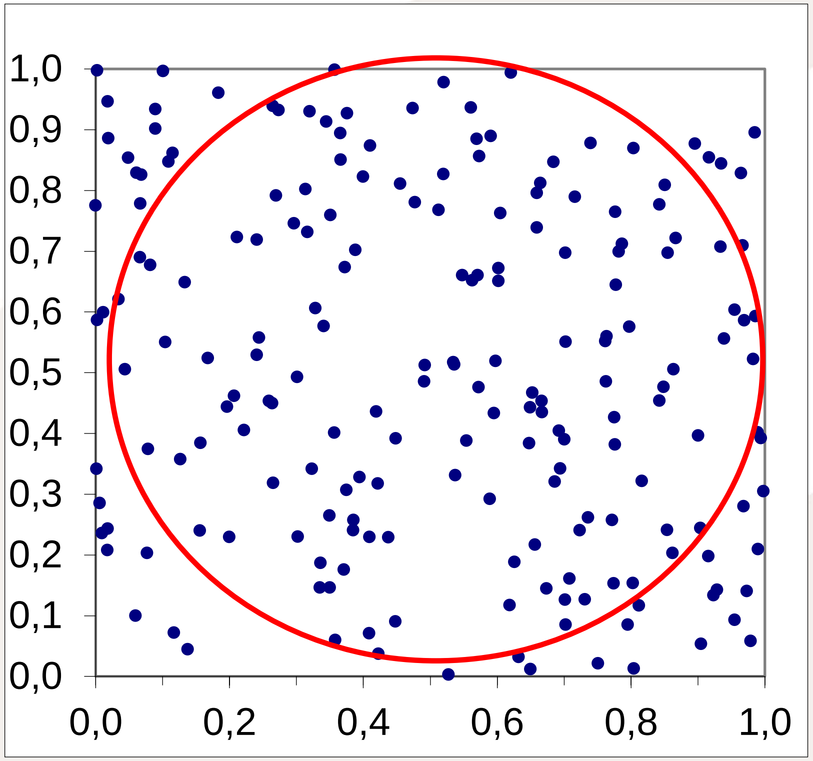
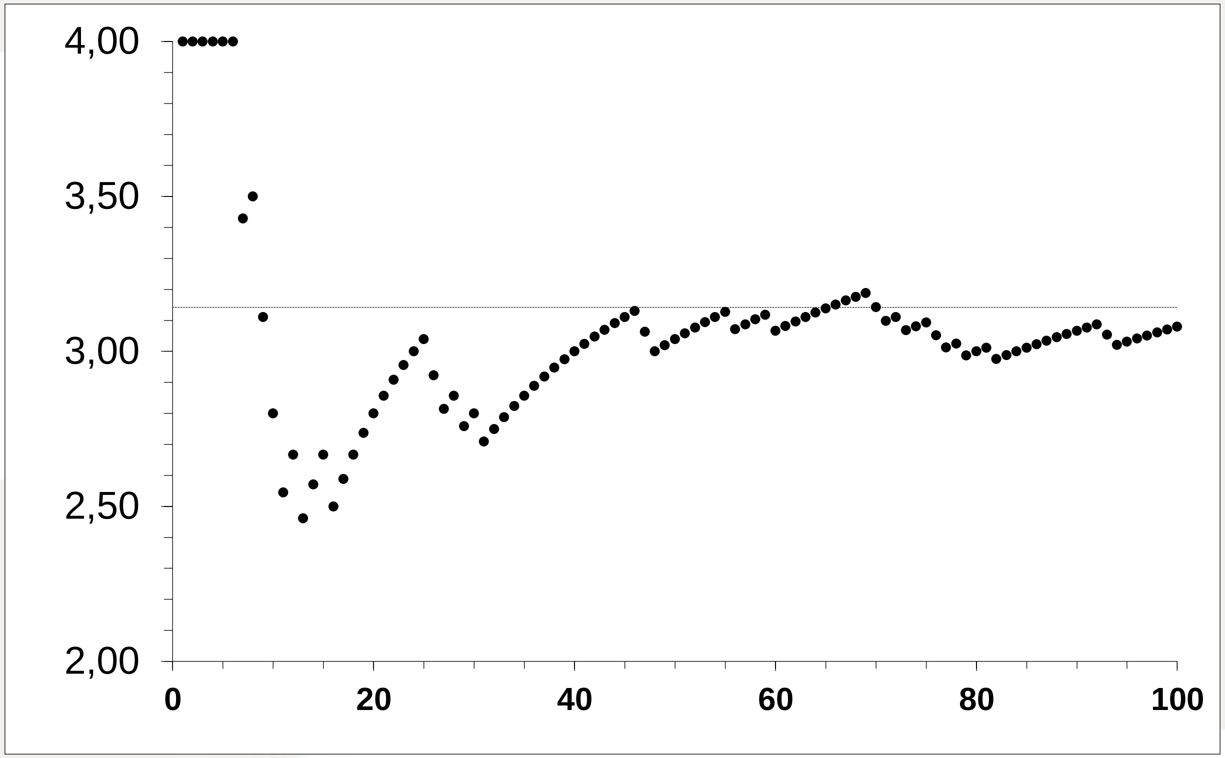
Monte-Carlo methods (evaluation of π)





$$\frac{N_{circle}}{N_{square}} \approx \frac{S_{circle}}{S_{square}} = \frac{\pi r^2}{1 \cdot 1} = \frac{\pi}{4} \rightarrow \pi \approx 4 \cdot \frac{N_{circle}}{N_{square}}$$

$N = 100$ $\pi = 3{,}08$
$N = 10^3$ $\pi = 3{,}204$
$N = 10^4$ $\pi = 3{,}155$
$N = 10^5$ $\pi = 3{,}1389$
$N = 10^6$ $\pi = 3{,}1427$
$N = 10^7$ $\pi = 3{,}1418$

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Monte-Carlo methods ((pseudo)random generator R[0,1])

Usually based on the remainder of integer divisions:

$x_{i+1} = a * x_i \bmod m$ $(i = 1, 2, \ldots)$ R[0,m-1]

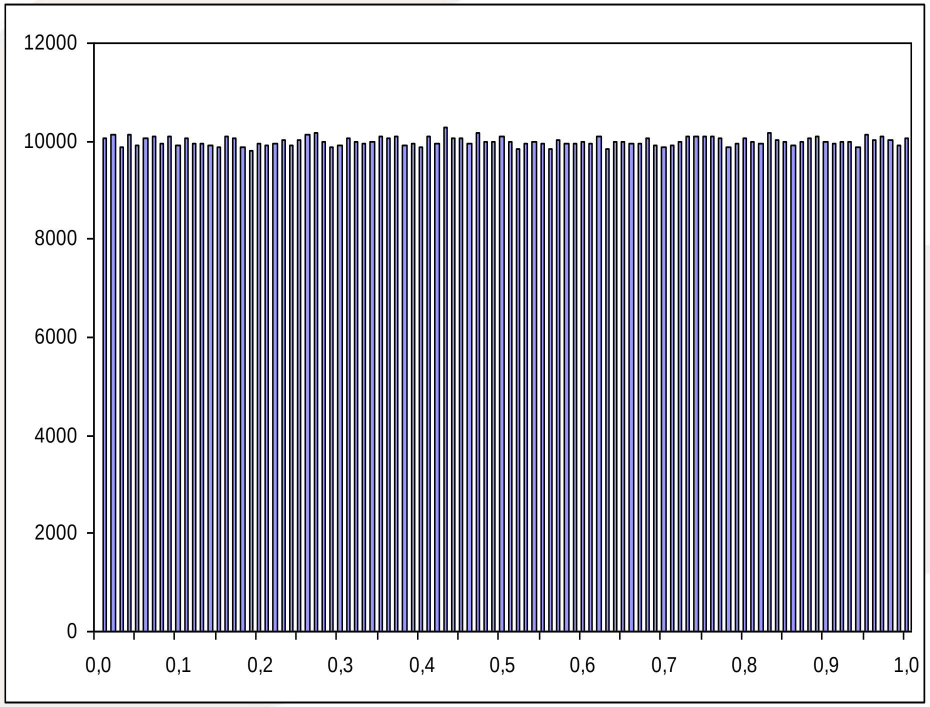
Example: $x_{i+1} = 7^5 * x_i \bmod (2^{31} - 1)$

12 mod 3 = 0 ↔ 12 = 3.4 + 0
11 mod 3 = 2 ↔ 11 = 3.3 + 2
10 mod 3 = 1 ↔ 10 = 3.3 + 1
9 mod 3 = 0 ↔ 9 = 3.3 + 0

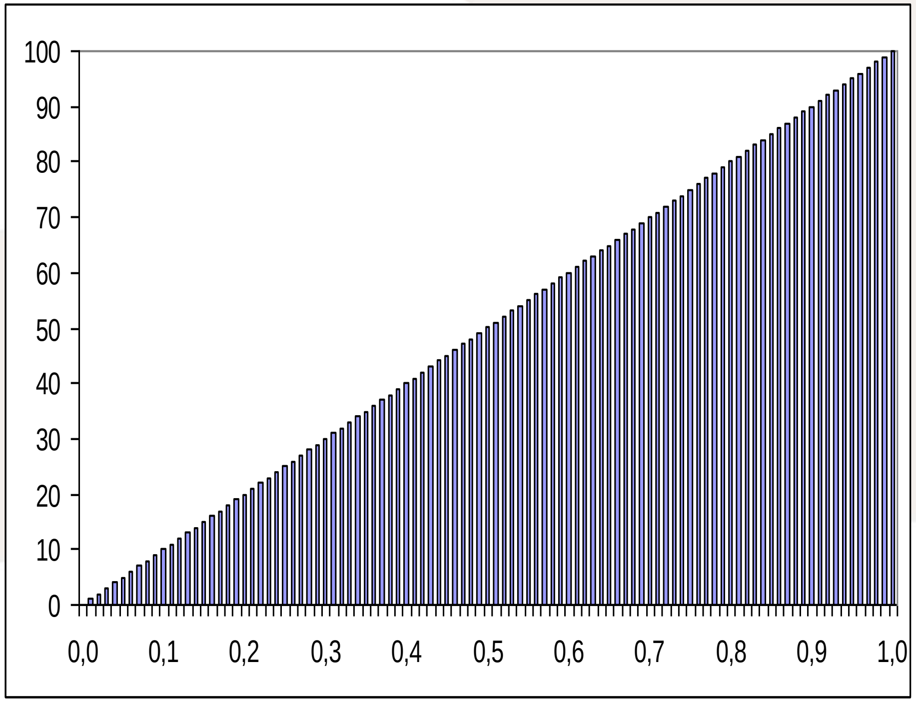
- The GUM+1 proposes a generator (enhanced Wichmann-Hill) that combines 4 R[0,m-1] generators. The proposed algorithm can be easily implemented.
- In the following examples, the random generator of VBA (Excel) has been used for practical reasons.

Monte-Carlo methods (Excel VBA generator R[0,1])

N = 10^6 function calls: Rnd()



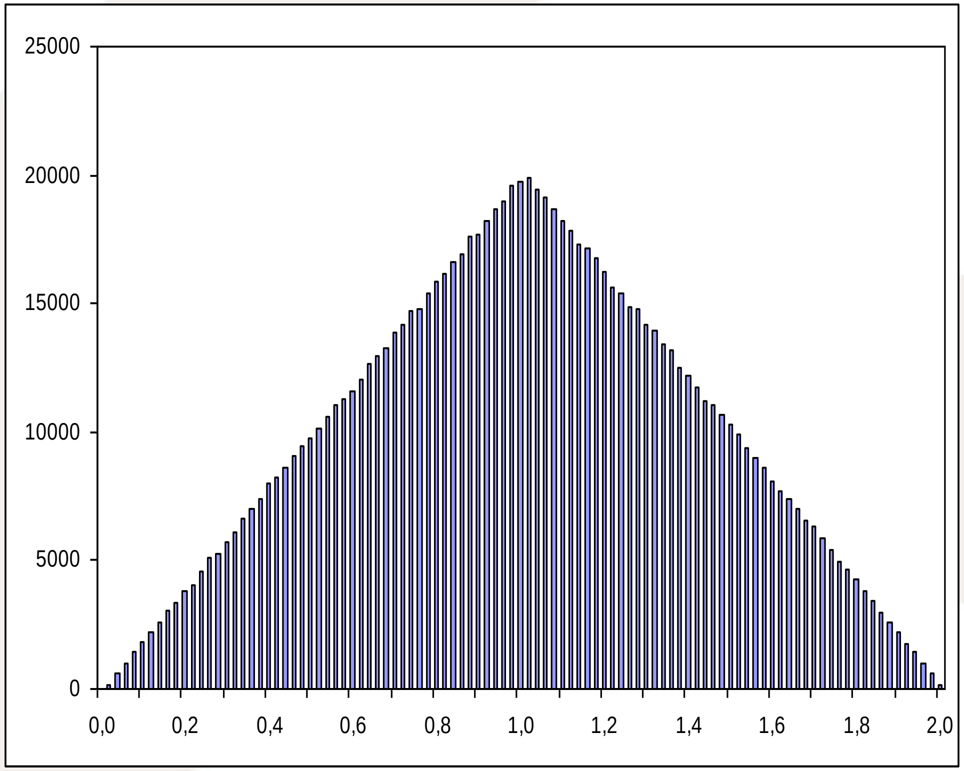
PDF



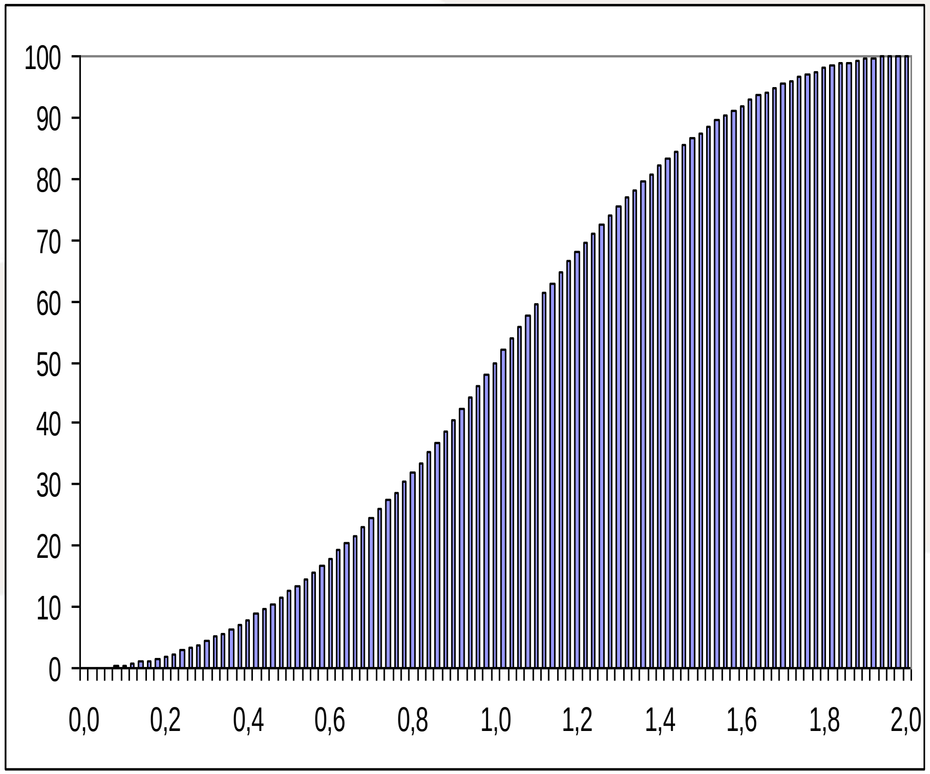
CDF (%)

Monte-Carlo methods (Excel VBA generator T[0,2])

N = 10^6 function calls: Rnd() + Rnd()



PDF

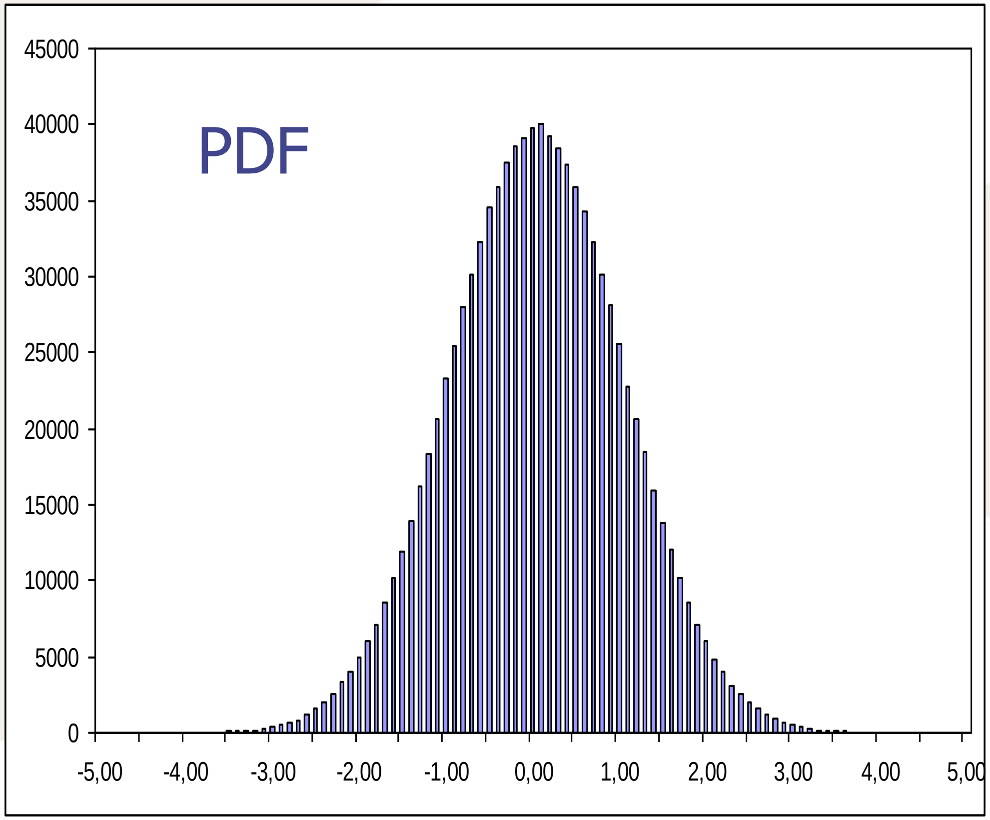


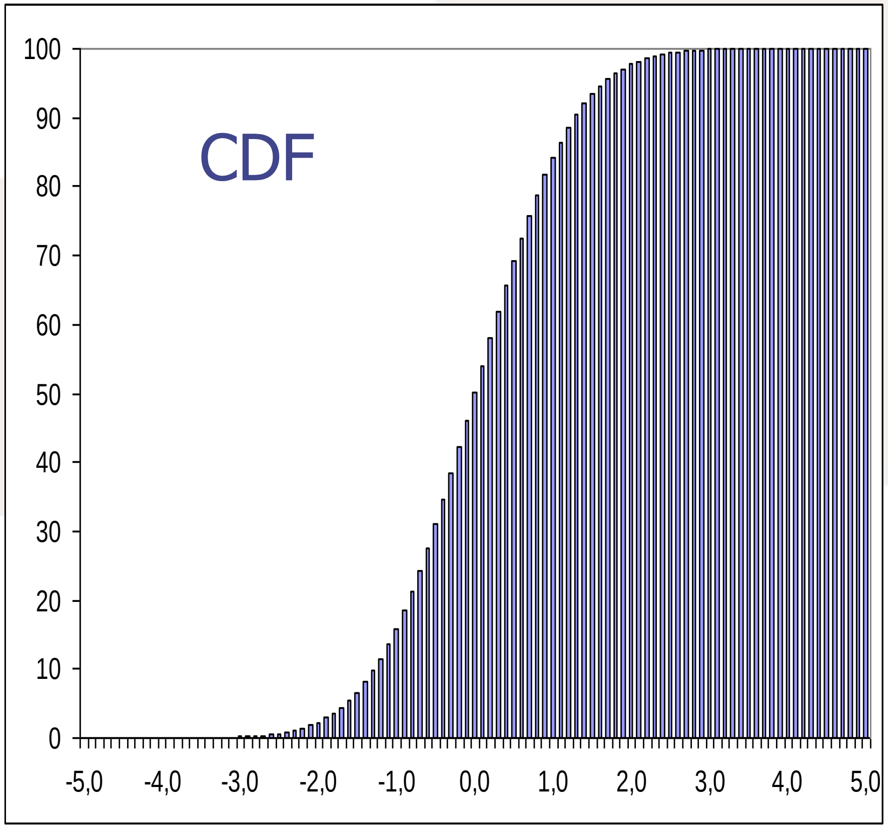
CDF (%)

Monte-Carlo methods (Excel VBA generator N[0,1])

N = 5.10^5 function calls: Box-Muller algorithm

r_1 = Rnd() : r_2 = Rnd(): $z_1 = (-2.\ln(r_1))^{1/2}.\cos(2.\pi.r_2)$: $z_2 = (-2.\ln(r_1))^{1/2}.\sin(2.\pi.r_2)$





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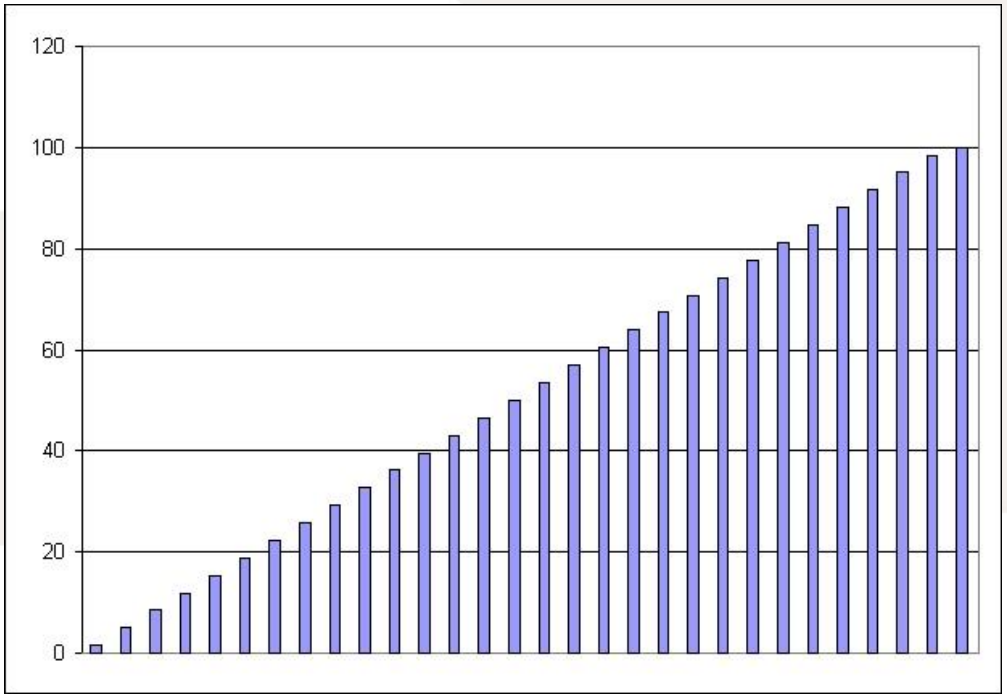
Enhanced Wichmann-Hill generator (C++)

N = 10^7 function calls

PDF



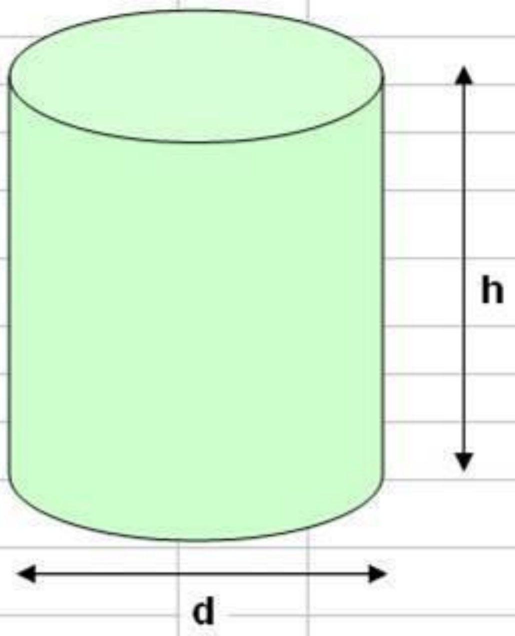
CDF (%)

Supplement 1 to the GUM

- 1. Comparison GUM – GUM+1
- 2. Monte-Carlo methods
- **3. Evaluation of measurement uncertainty using a MC method**
- 4. Calibration of a thermometer (GUM H3)

Evaluation of measurement uncertainty (1)

Calculation of the volume of a cylinder



Six determinations of diameter and height:

i	1	2	3	4	5	6
d_i /cm	4,985	5,000	5,020	4,975	4,980	5,005
h_i /cm	5,980	5,975	6,015	6,000	5,985	5,985

$\langle d \rangle$ =	4,994	cm
s_d =	0,017	cm
$s_{\langle d \rangle}$ =	0,007	cm

$\langle h \rangle$ =	5,990	cm
s_h =	0,015	cm
$s_{\langle h \rangle}$ =	0,006	cm

$$V = \pi . d^2 . h/4$$

$$V = 117,3391 \text{ cm}^3$$

Evaluation of measurement uncertainty (2)

UNCERTAINTIES				**Mean volume =**		**117,339**	**cm^3**
# iterations =		**1,00E+04**	**(≤ 10^7)**	**Progress =**		**1,00E+04**	
Start simulation				**Simulated volume =**		**117,3377**	**cm^3**
				Uncertainty (k=2) **=**		**0,7000**	**cm^3**
Quantity	**Unit**	**Distribution**	**Parameters**	**u_i (%)**	V_{min} =	116,653	cm^3
diameter	cm	normal	u = 0,007002	88,5769	V_{max} =	118,030	cm^3
height	cm	normal	u = 0,006055	11,4231	U (95%) =	0,689	cm^3
			∑ =	**100,00**			

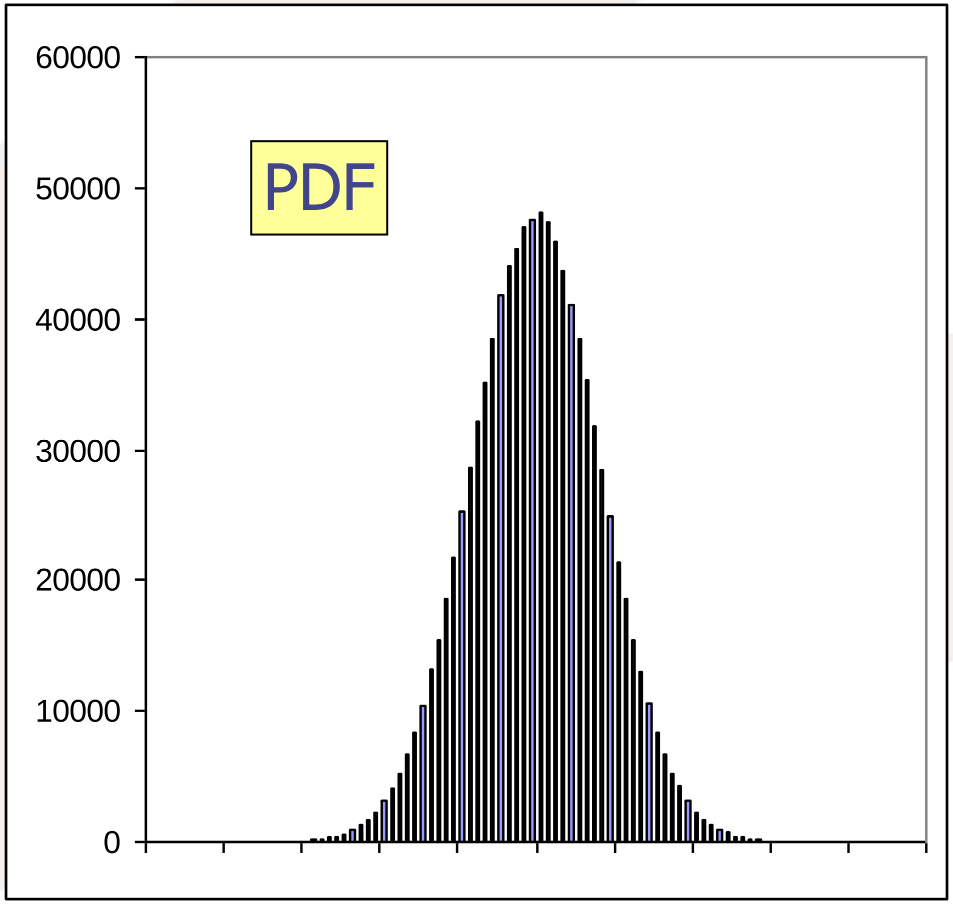
Program (1)

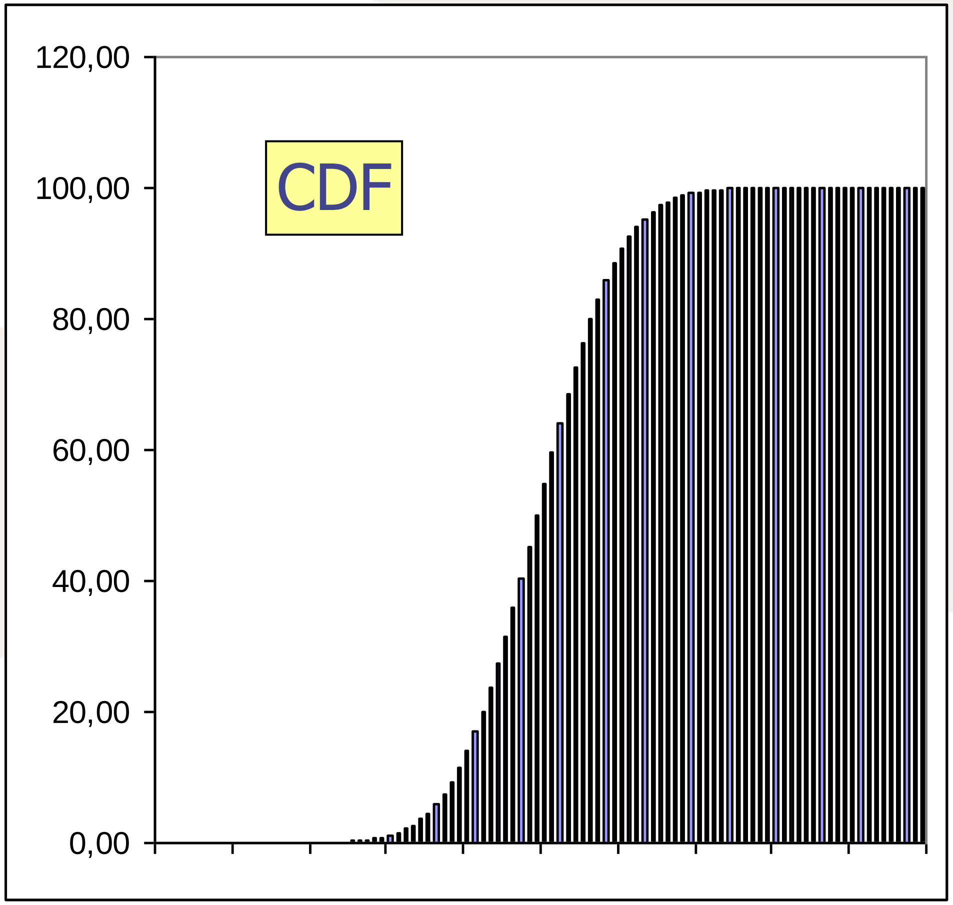
```
Dim i, N As Long: Dim Pi As Single : Dim r1, r2, d, h, V, Vmin, Vmax, Bin As Single : Dim Volumes(10000000#) As Single :
Dim DeltaV, AverageV, sV, DeltaVd, AverageVd, sVd, DeltaVh, AverageVh, sVh As Single
Cells(3, 7).Value = "": Cells(5, 7).Value = "": Cells(6, 7).Value = "": Cells(9, 5).Value = "": Cells(10, 5).Value = ""
Cells(8, 7).Value = "": Cells(9, 7).Value = "" : N = Cells(3, 3).Value: Pi = 3.141592654

For i = 1 To N: Cells(3, 7).Value = i
   'Uncertainty of V
   r1 = Rnd(): r2 = Rnd()
   d = Sqr(-2 * Log(r1)) * Cos(2 * Pi * r2) * Sheets("Data").Cells(14, 6).Value + Sheets("Data").Cells(12, 6).Value
   h = Sqr(-2 * Log(r1)) * Sin(2 * Pi * r2) * Sheets("Data").Cells(14, 10).Value + Sheets("Data").Cells(12, 10).Value
   V = Pi * d * d * h / 4: Volumes(i) = V
   DeltaV = V - AverageV: AverageV = AverageV + DeltaV / i: sV = sV + DeltaV * (V - AverageV)
   'Contribution of the diameter
   r1 = Rnd(): r2 = Rnd()
   d = Sqr(-2 * Log(r1)) * Cos(2 * Pi * r2) * Sheets("Data").Cells(14, 6).Value + Sheets("Data").Cells(12, 6).Value
   h = Sheets("Data").Cells(12, 10).Value
   V = Pi * d * d * h / 4
   DeltaVd = V - AverageVd: AverageVd = AverageVd + DeltaVd / i: sVd = sVd + DeltaVd * (V - AverageVd)
   'Contribution of the height
   r1 = Rnd(): r2 = Rnd()
   d = Sheets("Data").Cells(12, 6).Value
   h = Sqr(-2 * Log(r1)) * Sin(2 * Pi * r2) * Sheets("Data").Cells(14, 10).Value + Sheets("Data").Cells(12, 10).Value
   V = Pi * d * d * h / 4
   DeltaVh = V - AverageVh: AverageVh = AverageVh + DeltaVh / i: sVh = sVh + DeltaVh * (V - AverageVh)
Next i

'Display uncertainty and contributions
Cells(5, 7).Value = AverageV: Cells(6, 7).Value = Sqr(sV / (N - 1))
Cells(9, 5).Value = 100 * sVd / (N - 1) / Cells(6, 7).Value ^ 2: Cells(10, 5).Value = 100 * sVh / (N - 1) / Cells(6, 7).Value ^ 2
```

Program (2)

```
'Construction of PDF and CDF
Bin = 12 * Sqr(sV / (N - 1)) / 100
For i = 1 To 100
    Sheets("Histograms").Cells(i, 1).Value = (Cells(1, 7).Value - 6 * Sqr(sV / (N - 1))) + i * Bin
    Sheets("Histograms").Cells(i, 2).Value = ""   'Clear CDF cells
Next i
For i = 1 To N: Cells(3, 7).Value = i: j = 1
    While Volumes(i) > Sheets("Histograms").Cells(j, 1).Value: j = j + 1: Wend
    Sheets("Histograms").Cells(j, 2).Value = Sheets("Histograms").Cells(j, 2).Value + 1
Next i

'Search the uncertainty limits (CDF < 2.5 % and CDF > 97.5 %)
j = 1: While Sheets("Histograms").Cells(j, 3).Value < 2.5: j = j + 1: Wend
Vmin = Sheets("Histograms").Cells(j, 1) - Sheets("Histograms").Cells(j - 1, 1)
Vmin = Vmin * (2.5 - Sheets("Histograms").Cells(j - 1, 3))
Vmin = Vmin / (Sheets("Histograms").Cells(j, 3) - Sheets("Histograms").Cells(j - 1, 3))
Vmin = Vmin + Sheets("Histograms").Cells(j - 1, 1)
Cells(8, 7).Value = Vmin
j = 1: While Sheets("Histograms").Cells(j, 3).Value < 97.5: j = j + 1: Wend
Vmax = Sheets("Histograms").Cells(j, 1) - Sheets("Histograms").Cells(j - 1, 1)
Vmax = Vmax * (97.5 - Sheets("Histograms").Cells(j - 1, 3))
Vmax = Vmax / (Sheets("Histograms").Cells(j, 3) - Sheets("Histograms").Cells(j - 1, 3))
Vmax = Vmax + Sheets("Histograms").Cells(j - 1, 1)
Cells(9, 7).Value = Vmax

'Normalize contributions
Dim Total As Single

Total = Cells(9, 5).Value + Cells(10, 5).Value
Cells(9, 5).Value = Cells(9, 5).Value / Total * 100: Cells(10, 5).Value = Cells(10, 5).Value / Total * 100
```

Evaluation of measurement uncertainty (3)





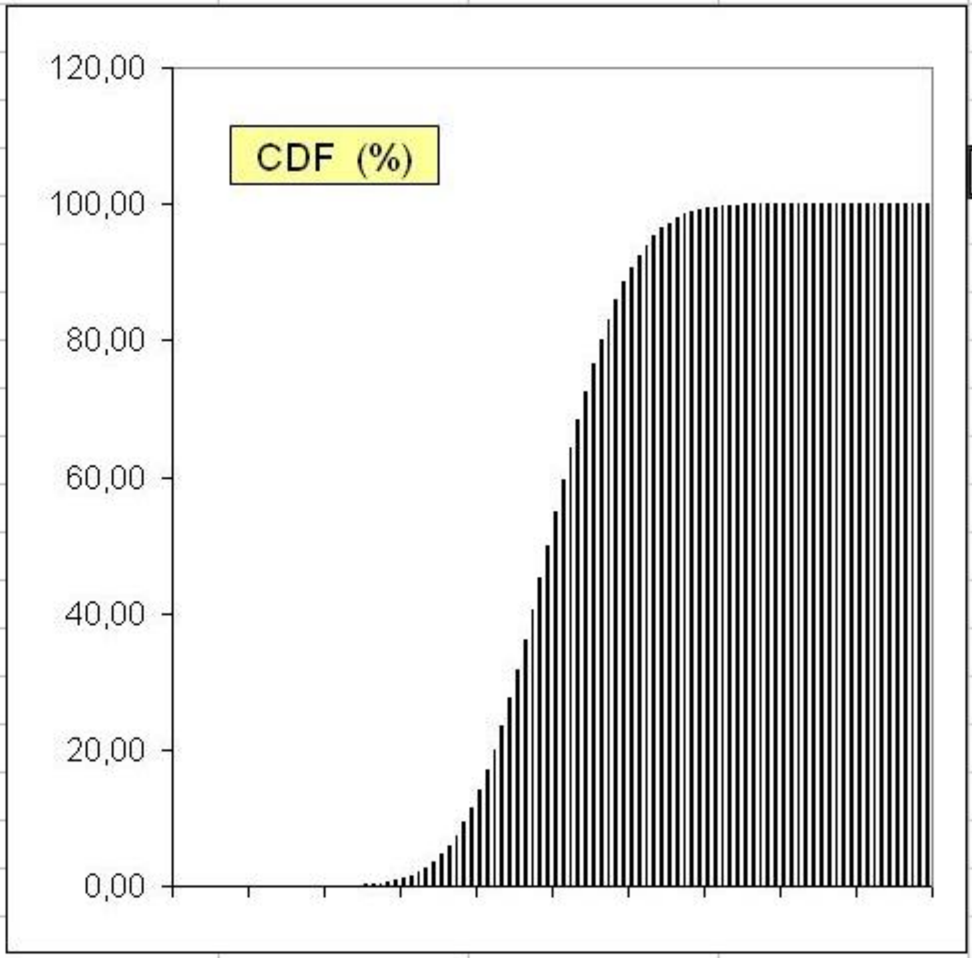
Evaluation of measurement uncertainty (4)

Bins

PDF

CDF

	Bins	PDF	CDF
21	116,121	88	0,02
22	116,163	127	0,04
23	116,205	189	0,05
24	116,247	320	0,09
25	116,289	406	0,13
26	116,331	633	0,19
27	116,373	894	0,28
28	116,415	1286	0,41
29	116,457	1764	0,58
30	116,499	2313	0,82
31	116,541	3099	1,13
32	116,583	4106	1,54
33	116,625	5155	2,05
34	116,667	6751	2,73
35	116,709	8292	3,56
36	116,751	10454	4,60
37	116,793	13097	5,91
38	116,835	15447	7,46
39	116,877	18630	9,32
40	116,919	21644	11,48
41	116,961	25234	14,01



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Evaluation of measurement uncertainty (5)

UNCERTAINTIES				**Mean volume =**		**117,339**	**cm³**
# iterations =	**1,00E+04**		**(≤ 10⁷)**	**Progress =**		**1,00E+04**	
Start simulation				**Simulated volume =**		**117,3377**	**cm³**
				Uncertainty (k=2) =		**0,7000**	**cm³**
Quantity	**Unit**	**Distribution**	**Parameters**	**u_i (%)**	V_{min} =	116,653	cm³
diameter	cm	normal	u = 0,007002	88,5769	V_{max} =	118,030	cm³
height	cm	normal	u = 0,006055	11,4231	U (95%) =	0,689	cm³
			∑ =	**100,00**			

118,030 cm^3 ≥ V ≥ 116,653 cm^3 (95 %)

Evaluation of measurement uncertainty (6)

COMPARISON GUM ↔ GUM+1

$$V = \pi . d^2 . h/4$$

$\langle h \rangle = 5{,}990$ cm
$s_h = 0{,}015$ cm
$s_{\langle h \rangle} = 0{,}006$ cm

$\langle d \rangle = 4{,}994$ cm
$s_d = 0{,}017$ cm
$s_{\langle d \rangle} = 0{,}007$ cm

$\partial V/\partial h = \pi . d^2/4 =$ 19,59 cm^4

$\partial V/\partial d = \pi . d . h/2 =$ 46,99 cm^4

Quantity	Uncertainty	Distribution	Sensitivity coefficient	u_i	Contributions		
height	0,006	normal, k = 1	19,59	0,119 cm^3	h	11,50	%
diameter	0,007	normal, k = 1	46,99	0,329 cm^3	d	88,50	%
			u (k=1)	0,350 cm^3			
			u (k=2)	**0,700 cm^3**			

Result GUM +1		
u =	0,69	cm^3
u_h =	11,42	%
u_d =	88,58	%

Evaluation of measurement uncertainty (7)

Comparison GUM ↔ GUM+1:

- GUM $u = 0{,}350\ \mathrm{cm}^3 \rightarrow U(k{=}2) = 0{,}700\ \mathrm{cm}^3$
- GUM+1 $U(95\%) = 0{,}694\ \mathrm{cm}^3$

The difference is due to the departure from a normal distribution of the PDF

Supplement 1 to the GUM

- 1. Comparison GUM – GUM+1
- 2. Monte-Carlo methods
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- **4. Calibration of a thermometer (GUM H3)**

Calibration of a thermometer GUM H3 (1)

Use of method of least squares to obtain linear calibration curve for a thermometer

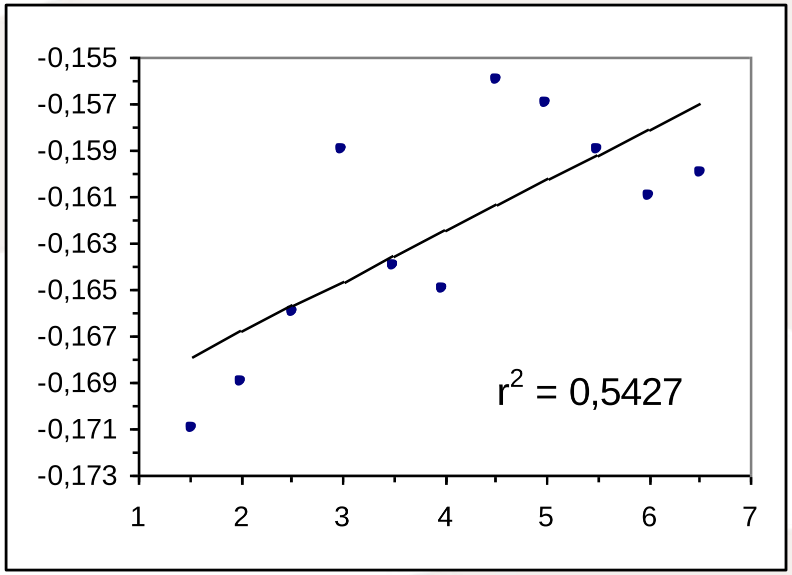
Mathematical model:

$$b(t) = y_1 + y_2.(t - t_0) \qquad (t_0 = 20\ °C)$$

Uncertainty of temperature measurements is negligible → uncertainty determined by uncertainty of parameters y_1 and y_2

Calibration of a thermometer GUM H3 (2)

k	t_k (°C)	t_k-20 (°C)	b_k (°C)
1	21,521	1,521	-0,171
2	22,012	2,012	-0,169
3	22,512	2,512	-0,166
4	23,003	3,003	-0,159
5	23,507	3,507	-0,164
6	23,999	3,999	-0,165
7	24,513	4,513	-0,156
8	25,002	5,002	-0,157
9	25,503	5,503	-0,159
10	26,010	6,010	-0,161
11	26,511	6,511	-0,160



$b(t_k)$ (°C)	b_k - $b(t_k)$ (°C)
-0,1679	-0,0031
-0,1668	-0,0022
-0,1657	-0,0003
-0,1646	0,0056
-0,1635	-0,0005
-0,1625	-0,0025
-0,1614	0,0054
-0,1603	0,0033
-0,1592	0,0002
-0,1581	-0,0029
-0,1570	-0,0030

y_1 = -0.1712 °C
$s(y_1)$ = 0.0029 °C

y_2 = 0.00218
$s(y_2)$ = 0.00067

$b(t) = y_1 + y_2.(t - t_0)$
$(t_0 = 20$ °C$)$

Input for the least squares

Input for the Monte Carlo simulations

Calibration of a thermometer GUM H3 (3)

Uncertainty of a predicted value from least-squares regression - Monte-Carlo method

Model: $b = y_1 + y_2.(t - 20)$ °C (no uncertainties on temperatures, GUM H.3 example)

data		simulation	Intermediate values		
t_i (°C)	b_i (°C)	b_i (°C)	$(x_i - x_m)^2$	$(x_i - x_m).(y_i - y_m)$	$(b_i - b_{i,c})^2$
21,521	-0,171				
22,012	-0,169				
22,512	-0,166				
23,003	-0,159				
23,507	-0,164				
23,999	-0,165				
24,513	-0,156				
25,002	-0,157				
25,503	-0,159				
26,010	-0,161				
26,511	-0,160				

y_1 =	
y_2 =	
$s_{y.x}$ =	

Start simulation

b =	
s(b) =	

bMin	
bMax	
u (*k* = 1)	
u (*k* = 2)	

Temperature for predicted error (°C): 30

How many iterations: 10000

Calibration of a thermometer GUM H3 (4)

```
'Declarations
Dim i, N As Long
Dim y1, y2, sy, sy1, sy2, t, r, r1, r2, b, Deltab, Meanb, sb, bMin, bMax As Single
Dim bValues(10000000#) As Single
Dim j As Integer

'Cleaning and initializing
Cells(13, 11).Value = "": Cells(14, 11).Value = ""
For j = 8 To 18: Cells(j, 4).Value = Cells(j, 2).Value: Next j
y1 = Cells(20, 18).Value
y2 = Cells(21, 18).Value
sy = Cells(23, 18).Value
sy1 = Cells(26, 18).Value
sy2 = Cells(27, 18).Value

'Input
N = Cells(22, 7).Value: t = Cells(20, 7).Value

'Iteration
For i = 1 To N: Cells(22, 7).Value = i
    r1 = Rnd() * 2 - 1
    r2 = Rnd() * 2 - 1
    For j = 8 To 18
        Cells(j, 4).Value = y1 + r1 * sy1 + (y2 + r2 * sy2) * (Cells(j, 1).Value - 20)
    Next j
    b = Cells(7, 11).Value + Cells(8, 11).Value * (t - 20)
    bValues(i) = b
    Deltab = b - Meanb: Meanb = Meanb + Deltab / i: sb = sb + Deltab * (b - Meanb)
Next i
```

Calibration of a thermometer GUM H3 (5)

```
'Output
Cells(13, 11).Value = Meanb: Cells(14, 11).Value = Sqr(sb / (N - 1))

'Construction of PDF and CDF
Bin = 12 * Sqr(sb / (N - 1)) / 100
For i = 1 To 100
    Sheets("Histograms").Cells(i, 1).Value = (Cells(13, 11).Value - 6 * Sqr(sb / (N - 1))) + i * Bin
    Sheets("Histograms").Cells(i, 2).Value = ""
Next i
For i = 1 To N: Cells(3, 7).Value = i: j = 1
    While bValues(i) > Sheets("Histograms").Cells(j, 1).Value: j = j + 1: Wend
    Sheets("Histograms").Cells(j, 2).Value = Sheets("Histograms").Cells(j, 2).Value + 1
Next i

'Search the uncertainty limits  (CDF < 2.5 % and CDF > 97.5 %)
j = 1: While Sheets("Histograms").Cells(j, 3).Value < 2.5: j = j + 1: Wend
bMin = Sheets("Histograms").Cells(j, 1) - Sheets("Histograms").Cells(j - 1, 1)
bMin = bMin * (2.5 - Sheets("Histograms").Cells(j - 1, 3))
bMin = bMin / (Sheets("Histograms").Cells(j, 3) - Sheets("Histograms").Cells(j - 1, 3))
bMin = bMin + Sheets("Histograms").Cells(j - 1, 1)
Cells(13, 14).Value = bMin
j = 1: While Sheets("Histograms").Cells(j, 3).Value < 97.5: j = j + 1: Wend
bMax = Sheets("Histograms").Cells(j, 1) - Sheets("Histograms").Cells(j - 1, 1)
bMax = bMax * (97.5 - Sheets("Histograms").Cells(j - 1, 3))
bMax = bMax / (Sheets("Histograms").Cells(j, 3) - Sheets("Histograms").Cells(j - 1, 3))
bMax = bMax + Sheets("Histograms").Cells(j - 1, 1)
Cells(14, 14).Value = bMax
```

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Calibration of a thermometer GUM H3 (6)

Uncertainty of a predicted value from least-squares regression - Monte-Carlo method

Model: $b = y_1 + y_2.(t - 20)$ °C (no uncertainties on temperatures, GUM H.3 example)

data		simulation	Intermediate values		
t_i (°C)	b_i (°C)	b_i (°C)	$(x_i - x_m)^2$	$(x_i - x_m).(y_i - y_m)$	$(b_i - b_{i,c})^2$
21,521	-0,171	-0,166	6,19E+00	1,34E-02	0,00E+00
22,012	-0,169	-0,165	3,99E+00	8,62E-03	1,47E-34
22,512	-0,166	-0,164	2,24E+00	4,84E-03	1,08E-34
23,003	-0,159	-0,163	1,01E+00	2,19E-03	7,52E-37
23,507	-0,164	-0,161	2,51E-01	5,44E-04	7,52E-37
23,999	-0,165	-0,160	8,94E-05	1,93E-07	1,08E-34
24,513	-0,156	-0,159	2,55E-01	5,51E-04	4,81E-35
25,002	-0,157	-0,158	9,87E-01	2,14E-03	3,01E-36
25,503	-0,159	-0,157	2,23E+00	4,83E-03	0,00E+00
26,010	-0,161	-0,156	4,01E+00	8,67E-03	1,20E-35
26,511	-0,160	-0,155	6,26E+00	1,35E-02	4,81E-35

y_1 =	-0,16900
y_2 =	0,00216
$s_{y.x}$ =	0,00000
b =	-0,1494
s(b) =	0,0042

Start simulation

bMin	-0,15698795
bMax	-0,14186443
u (*k* =1)	0,00378088
u (*k* =2)	0,00756176

Temperature for predicted error (°C): 30

How many iterations: 10000

GUM		GUM + 1 (VBA)		GUM + 1 (C++)
b = - 0,149 4 °C	⟷	b = - 0,149 4 °C	⟷	b = - 0,149 4 °C
s(b) = 0,004 1 °C		s(b) = 0,004 2 °C		s(b) = 0,004 2 °C
		u(*k*=2) = 0,007 6 °C		u(*k*=2) = 0,007 6 °C

JCGM 101:2008

Evaluation of measurement data — Supplement 1 to the "Guide to the expression of uncertainty in measurement" — Propagation of distributions using a Monte Carlo method

Évaluation des données de mesure — Supplément 1 du "Guide pour l'expression de l'incertitude de mesure" — Propagation de distributions par une méthode de Monte Carlo