

If $z^n = u_n + iv_n$, then $z^{n+1} = u_{n+1} + iv_{n+1}$ where

3.7.23 $u_{n+1} = xu_n - yv_n$; $v_{n+1} = xv_n + yu_n$
 $\mathcal{R}z^n$ and $\mathcal{I}z^n$ are called harmonic polynomials.

3.7.24
$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$$

3.7.25
$$\frac{1}{z^n} = \frac{\bar{z}^n}{|z|^{2n}} = (z^{-1})^n$$

Roots

3.7.26 $z^{\frac{1}{n}} = \sqrt[n]{z} = r^{\frac{1}{n}} e^{i\theta/n} = r^{\frac{1}{n}} \cos \frac{\theta}{n} + ir^{\frac{1}{n}} \sin \frac{\theta}{n}$

If $-\pi < \theta \leq \pi$ this is the principal root. The other root has the opposite sign. The principal root is given by

3.7.27 $z^{\frac{1}{n}} = [\frac{1}{2}(r+x)]^{\frac{1}{n}} \pm i[\frac{1}{2}(r-x)]^{\frac{1}{n}} = u \pm iv$ where $2uv = y$ and where the ambiguous sign is taken to be the same as the sign of y .

3.7.28 $z^{1/n} = r^{1/n} e^{i\theta/n}$, (principal root if $-\pi < \theta \leq \pi$). Other roots are $r^{1/n} e^{i(\theta+2\pi k)/n}$ ($k=1, 2, 3, \dots, n-1$).

Inequalities

3.7.29
$$\left| |z_1| - |z_2| \right| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Complex Functions, Cauchy-Riemann Equations

$f(z) = f(x+iy) = u(x, y) + iv(x, y)$ where $u(x, y), v(x, y)$ are real, is *analytic* at those points $z = x+iy$ at which

3.7.30
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If $z = re^{i\theta}$,

3.7.31
$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

Laplace's Equation

The functions $u(x, y)$ and $v(x, y)$ are called harmonic functions and satisfy Laplace's equation:

Cartesian Coordinates

3.7.32
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Polar Coordinates

3.7.33
$$r \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial \theta^2} = r \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{\partial^2 v}{\partial \theta^2} = 0$$

3.8. Algebraic Equations

Solution of Quadratic Equations

3.8.1 Given $az^2 + bz + c = 0$,

$$z_{1,2} = -\left(\frac{b}{2a}\right) \pm \frac{1}{2a} q^{\frac{1}{2}}, \quad q = b^2 - 4ac,$$

$$z_1 + z_2 = -b/a, \quad z_1 z_2 = c/a$$

If $q > 0$, two real roots,
 $q = 0$, two equal roots,
 $q < 0$, pair of complex conjugate roots.

Solution of Cubic Equations

3.8.2 Given $z^3 + a_2 z^2 + a_1 z + a_0 = 0$, let

$$q = \frac{1}{3} a_1 - \frac{1}{9} a_2^2; \quad r = \frac{1}{6} (a_1 a_2 - 3a_0) - \frac{1}{27} a_2^3.$$

If $q^3 + r^2 > 0$, one real root and a pair of complex conjugate roots,

$q^3 + r^2 = 0$, all roots real and at least two are equal,

$q^3 + r^2 < 0$, all roots real (irreducible case).

Let

$$s_1 = [r + (q^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}}, \quad s_2 = [r - (q^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}}$$

then

$$z_1 = (s_1 + s_2) - \frac{a_2}{3}$$

$$z_2 = -\frac{1}{2} (s_1 + s_2) - \frac{a_2}{3} + \frac{i\sqrt{3}}{2} (s_1 - s_2)$$

$$z_3 = -\frac{1}{2} (s_1 + s_2) - \frac{a_2}{3} - \frac{i\sqrt{3}}{2} (s_1 - s_2).$$

If z_1, z_2, z_3 are the roots of the cubic equation

$$z_1 + z_2 + z_3 = -a_2$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = a_1$$

$$z_1 z_2 z_3 = -a_0$$

Solution of Quartic Equations

3.8.3 Given $z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$, find the real root u_1 of the cubic equation

$$u^3 - a_2 u^2 + (a_1 a_3 - 4a_0) u - (a_1^2 + a_0 a_3^2 - 4a_0 a_2) = 0$$

and determine the four roots of the quartic as solutions of the two quadratic equations

$$v^2 + \left[\frac{a_3}{2} \mp \left(\frac{a_3^2}{4} + u_1 - a_2 \right)^{\frac{1}{2}} \right] v + \frac{u_1}{2} \mp \left[\left(\frac{u_1}{2} \right)^2 - a_0 \right]^{\frac{1}{2}} = 0$$

If all roots of the cubic equation are real, use the value of u_1 which gives real coefficients in the *quadratic equation and select signs so that if

$$z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 = (z^2 + p_1z + q_1)(z^2 + p_2z + q_2),$$

then

$$p_1 + p_2 = a_3, p_1p_2 + q_1 + q_2 = a_2, p_1q_2 + p_2q_1 = a_1, q_1q_2 = a_0.$$

If z_1, z_2, z_3, z_4 are the roots,

$$\Sigma z_i = -a_3, \Sigma z_i z_j z_k = -a_1,$$

$$\Sigma z_i z_j = a_2, z_1 z_2 z_3 z_4 = a_0.$$

3.9. Successive Approximation Methods

General Comments

3.9.1 Let $x = x_1$ be an approximation to $x = \xi$ where $f(\xi) = 0$ and both x_1 and ξ are in the interval $a \leq x \leq b$. We define

$$x_{n+1} = x_n + c_n f(x_n) \quad (n = 1, 2, \dots).$$

Then, if $f'(x) \geq 0$ and the constants c_n are negative and bounded, the sequence x_n converges monotonically to the root ξ .

If $c_n = c = \text{constant} < 0$ and $f'(x) > 0$, then the process converges but not necessarily monotonically.

Degree of Convergence of an Approximation Process

3.9.2 Let x_1, x_2, x_3, \dots be an infinite sequence of approximations to a number ξ . Then, if

$$|x_{n+1} - \xi| < A|x_n - \xi|^k, \quad (n = 1, 2, \dots)$$

where A and k are independent of n , the sequence is said to have convergence of at most the k th degree (or order or index) to ξ . If $k = 1$ and $A < 1$ the convergence is linear; if $k = 2$ the convergence is quadratic.

Regula Falsi (False Position)

3.9.3 Given $y = f(x)$ to find ξ such that $f(\xi) = 0$, choose x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ have opposite signs and compute

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} f_1 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}.$$

Then continue with x_2 and either of x_0 or x_1 for which $f(x_0)$ or $f(x_1)$ is of opposite sign to $f(x_2)$.

Regula falsi is equivalent to inverse linear interpolation.

*See page 11.

Method of Iteration (Successive Substitution)

3.9.4 The iteration scheme $x_{k+1} = F(x_k)$ will converge to a zero of $x = F(x)$ if

$$(1) |F'(x)| \leq q < 1 \text{ for } a \leq x \leq b,$$

$$(2) a \leq x_0 \pm \frac{|F(x_0) - x_0|}{1 - q} \leq b.$$

Newton's Method of Successive Approximations

3.9.5

Newton's Rule

If $x = x_k$ is an approximation to the solution $x = \xi$ of $f(x) = 0$ then the sequence

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

will converge quadratically to $x = \xi$: (if instead of the condition (2) above),

(1) *Monotonic convergence*, $f(x_0)f''(x_0) > 0$ and $f'(x), f''(x)$ do not change sign in the interval (x_0, ξ) , or

(2) *Oscillatory convergence*, $f(x_0)f''(x_0) < 0$ and $f'(x), f''(x)$ do not change sign in the interval (x_0, x_1) , $x_0 \leq \xi \leq x_1$.

Newton's Method Applied to Real n th Roots

3.9.6 Given $x^n = N$, if x_k is an approximation $x = N^{1/n}$ then the sequence

$$x_{k+1} = \frac{1}{n} \left[\frac{N}{x_k^{n-1}} + (n-1)x_k \right]$$

will converge quadratically to x .

$$\text{If } n = 2, x_{k+1} = \frac{1}{2} \left(\frac{N}{x_k} + x_k \right),$$

$$\text{If } n = 3, x_{k+1} = \frac{1}{3} \left(\frac{N}{x_k^2} + 2x_k \right).$$

Aitken's δ^2 -Process for Acceleration of Sequences

3.9.7 If x_k, x_{k+1}, x_{k+2} are three successive iterates in a sequence converging with an error which is approximately in geometric progression, then

$$\bar{x}_k = x_k - \frac{(x_k - x_{k+1})^2}{\Delta^2 x_k} = \frac{x_k x_{k+2} - x_{k+1}^2}{\Delta^2 x_k},$$

$$\Delta^2 x_k = x_k - 2x_{k+1} + x_{k+2}$$

is an improved estimate of x . In fact, if $x_k = x + O(\lambda^k)$ then $\bar{x}_k = x + O(\lambda^{2k})$, $|\lambda| < 1$.