Lambda Theory

Introduction of a constant for the Void in Set Theory

Definition of Pre-element

- x is an element $\Leftrightarrow$ x is a set
- x is a pre-element $\Leftrightarrow$ $\forall$y(y is an element $\Rightarrow$ x $\in$ y)
- x is a set $\Leftrightarrow$ $\exists$y(y $\in$ x)

Lambda Language

- Logical symbols

 - an infinite set of variable symbols: x, y, z…

 - logical constants: $\neg$, $\wedge$, $\vee$, $\Rightarrow$

 - quantifiers symbols: $\forall$, $\exists$

 - parenthesis: (,)

 - equality symbol: =

Lambda Language

- Non-logical symbols

- symbol of constant Λ denoting the void
- a set of symbols of constants for sets
- an infinite set of function symbols of arity ≥ 0: f, g, h…
- only one relation symbol « $\in$ » of arity 2 for membership

Formation Rules: Terms

- Any constant, including Lambda, is a term.
- Any variable is a term.
- Any expression $f(t_1,\ldots,t_n)$ of n $\geq$ 1 arguments (where each argument t_i is a term and f is a function symbol of valence n) is a term.
- Closure clause: nothing else is a term.

Well-formed Formulas

- $x \in y$ and $x = y$ are atomic formulas
- Simple and complex predicates: if P is a relation of valence ≥ 0, and the a_i are terms, then $P(a_1,\ldots,a_n)$ is well-formed. If equality is considered as a part of the logic, then $(a_1 = a_2)$ is a wff.

Inductive Clauses

- Inductive clause 1: if φ is a wff, then $\neg\varphi$ is a wff.
- Inductive clause 2: if φ and ψ are wff, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$ are wff.
- Inductive clause 3: if φ is a wff, and x is a variable, then $\forall x\ \varphi$, $\exists x\ \varphi$ are wff.
- Closure Clause: nothing else is a wff.
- A sentence is a wff without variable of any sort.

Axiom of Pre-element

$$\forall x (x \neq \Lambda \Rightarrow \Lambda \in x)$$

Additional semantical definitions

- $\forall x(\neg(x \in \Lambda))$
- $\forall x((x = \Lambda) \vee (\Lambda \in x))$

Axiom of Pre-element II

$$\forall x(x \neq \Lambda \Leftrightarrow \Lambda \in x)$$

Axiom of Lambda

$$\exists x \forall y (\neg (y \in x))$$

$$x = \Lambda$$

Unchanged Standard Axioms

- Extensionality: $\forall x \forall y(\forall z(z \in x \Leftrightarrow z \in y \Rightarrow x = y))$
- Pairing: $\forall x \forall y \exists z(\forall t(t \in z \Rightarrow (t = x \vee t = y)))$
- Union: $\forall x \exists y \forall u(u \in y \Rightarrow \exists z(z \in x \wedge u \in z))$
- Power Set: $\forall x \exists y(\forall z(\forall t(t \in z \Rightarrow t \in x) \Rightarrow z \in y))$

Modified Standard Axioms

- Separation Scheme: $\forall p_1,\ldots,p_n \forall x \exists y \forall z(z \in y \Leftrightarrow (z \in x \wedge (\Phi(z, p_1,\ldots,p_n) \vee z = \Lambda)))$
- Replacement Scheme: $\forall p_1,\ldots,p_n (\forall x \forall y \forall z((\Phi(x, y, p_1,\ldots,p_n) \wedge \Phi(x, z, p_1,\ldots,p_n)) \Rightarrow y = z) \Rightarrow \forall x \exists y \forall z(z \in y \Leftrightarrow \exists u(u \in x \wedge (\Phi(u, z, p_1,\ldots,p_n)) \vee z = \Lambda)))$
- Regularity: $\forall x(\exists y(y \neq \Lambda \wedge y \in x \Rightarrow \exists y(y \neq \Lambda \wedge y \in x \wedge \neg \exists z(z \neq \Lambda \wedge z \in y \wedge z \in x)))$

Standard Axiom of Infinite

$$\exists x(\varnothing \in x \wedge \forall y(y \in x \Rightarrow y \cup \{y\} \in x))$$

Lambda Axiom of Infinite

$$\exists x \forall y (y \in x \Rightarrow y \cup \{y\} \in x)$$

Empty Set Theorem

$$\wp(\Lambda) = \{\Lambda\}$$

$$\forall x \exists y (\forall z (\forall t (t \in z \Rightarrow t \in x) \Rightarrow z \in y))$$

Empty Set by the Axiom of Pairing

$$\forall x \forall y \exists z(\forall t(t \in z \Rightarrow (t = x \vee t = y)))$$

With $x = y = \Lambda$, we have $z = \{\Lambda\}$

Lambda and Singleton of Lambda

- $\exists x \forall y(\neg(y \in x))$
- $\exists x \forall y(y \in x \Rightarrow y = \Lambda)$
- $x = \Lambda$
- $x = \{\Lambda\}$

Lambda and Contradictory Property

- What about the standard definition of empty set by means of a contradictory property?

$$\{x : x \neq x\}$$

- By the axiom of pre-element, Λ must belong to this set
- $\{x : x \neq x\} = \{\Lambda\}$
- $(x = x \Leftrightarrow x \neq x) \Rightarrow x$ is a pre-element

Singleton of Lambda and Standard Empty Set

- Do contain nothing $\equiv$ do not contain anything
- Free of sets ($\{\Lambda\}$) $\equiv$ free of sets and Lambda ($\varnothing$)

???

- Either the singleton of Lambda is the revised version of the standard empty set, or Set theory does not need empty set.

Behaviour of Singleton of Lambda and of Standard Empty Set

With x and y $\neq \{\Lambda\}$ and no common element

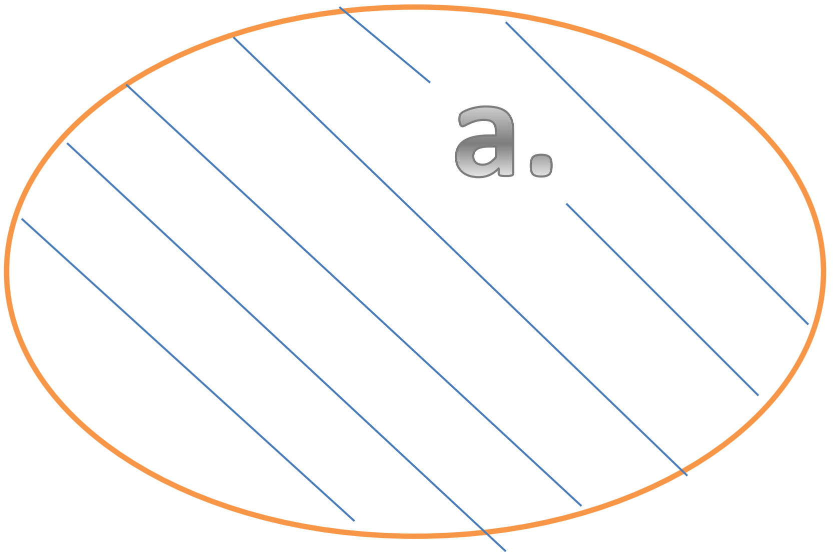
Singleton of Lambda

- $x \cap y = \{\Lambda\}$
- $x \cap \{\Lambda\} = \{\Lambda\}$
- $x \cup \{\Lambda\} = x$
- $x \setminus \{\Lambda\} = x$
- $x \setminus x = \{\Lambda\}$
- $\{\Lambda\} \cap \{\Lambda\} = \{\Lambda\}$
- $\{\Lambda\} \cup \{\Lambda\} = \{\Lambda\}$
- $\{\Lambda\} \setminus \{\Lambda\} = \{\Lambda\}$

Standard Empty Set

- $x \cap y = \varnothing$
- $x \cap \varnothing = \varnothing$
- $x \cup \varnothing = x$
- $x \setminus \varnothing = x$
- $x \setminus x = \varnothing$
- $\varnothing \cap \varnothing = \varnothing$
- $\varnothing \cup \varnothing = \varnothing$
- $\varnothing \setminus \varnothing = \varnothing$

Representation of Lambda



Behaviour of Lambda

- Lambda is the non numerical form of zero, or zero is the numerical form of Lambda, i.e. of the absence.
- $x \cap y = \{\Lambda\}$
- $x \cap \Lambda = \{\Lambda\}$
- $x \cup \Lambda = x$
- $x \setminus \Lambda = x$
- $\Lambda \cap \Lambda = \{\Lambda\}$
- $\Lambda \cup \Lambda = \{\Lambda\}$
- $\Lambda \setminus \Lambda = \{\Lambda\}$

Technical Interest of Lambda

- New definition of Set
- New definition of Empty Set
- No need of a contradictory property
- Uselessness of Empty Set Axiom and of Axiom of Existence of Set
- Simplification of the Axiom of Infinite
- Distinction empty set/ur-element

Conceptual Interest of Lambda

- Solution to the Puzzle of the Null Class (Russell)
- Distinction Empty set/void/zero (Lambda is not the empty set because it is not a set)

Future Developments

- Logic and theory of Potential
- Solution to empty family intersection anomaly

Questions

- L'axiome d'existence de Lambda que j'ai ajouté vous semble-t-il nécessaire?
- Que pensez-vous de l'appartenance de Lambda à l'ensemble construit au moyen de la définition contradictoire? Vous pose-t-elle un problème?
- Que pensez-vous du fait qu'on ne puisse soustraire Lambda à un ensemble dans une différence symétrique? Cela vous pose-t-il un problème?

- Que pensez-vous de la construction du Singleton de Lambda au moyen de l’axiome de l’ensemble des parties?
- Que pensez-vous de l’étude du comportement de Lambda dans des opérations comme l’intersection, l’union…?