# Lambda Theory

Introduction of a constant for the Void in Set Theory

# Lambda Language

- Logical symbols

- an infinite set of variable symbols: *x*, *y*, *z*…

- logical constants: $\neg$, $\wedge$, $\vee$, $\Rightarrow$

- quantifiers symbols: $\forall$, $\exists$

- parenthesis: (, )

- equality symbol: =

# Lambda Language

- Non-logical symbols

  - a symbol of constant denoting the void: $\Lambda$

  - an infinite set of function symbols of arity ≥ 0 denoted by lowercase letters *f, g*, *h*…

  - function symbols of valence 0 are called **constant symbols**, and are denoted by lowercase letters at the beginning of the alphabet *a*, *b*, *c*,...

  - only one relation symbol « $\in$ » of arity 2 for membership

- The signature $\sigma = (\Lambda, \in)$

# Formation Rules: Terms

- Any constant, including Lambda, is a term.
- Any variable is a term.
- Any expression $f(t_1,\ldots,t_n)$ of n $\geq$ 1 arguments (where each argument $t_i$ is a term and $f$ is a function symbol of valence $n$) is a term.
- Closure clause: nothing else is a term.

# Well-formed Formulas

- $x \in y$ and $x = y$ are atomic formulas
- Simple and complex predicates: if P is a relation of valence $\geq 0$, and the $a_i$ are terms, then $P(a_1,\ldots,a_n)$ is well-formed. Since equality is considered as a part of the logic, $(a_1 = a_2)$ is a wff.

# Inductive Clauses

- Inductive clause 1: if $\varphi$ is a wff, then $\neg\varphi$ is a wff.
- Inductive clause 2: if $\varphi$ and $\psi$ are wff, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$ are wff.
- Inductive clause 3: if $\varphi$ is a wff, and x is a variable, then $\forall x\ \varphi$, $\exists x\ \varphi$ are wff.
- Closure Clause: nothing else is a wff.
- A sentence is a wff without variable of any sort.

# Axiom of Pre-element or Axiom of Condition

$$\forall x(x \neq \Lambda \Rightarrow \Lambda \in x)$$

# Additional semantical definitions

The first additional definition implies that Lambda does not belong to Lambda. Consequently, Lambda is not a set.

The second definition is the contraposition of the axiom of pre-element.

- $\forall x(\neg(x \in \Lambda))$
- $\forall x((x = \Lambda) \vee (\Lambda \in x))$

# Axiom of Pre-element II

$$\forall x (x \neq \Lambda \Leftrightarrow \Lambda \in x)$$

Because of: $\forall x(\neg(x \in \Lambda))$

# Definition of Element, Pre-element and Set

- x is an element $\Leftrightarrow$ x is a set

- x is a pre-element $\Leftrightarrow$ $\forall$y(y is an element $\Rightarrow$ x $\in$ y)

- x is a set $\Leftrightarrow$ $\exists$y(y $\in$ x)

# Uselessness of Axiom of Lambda

$\exists x \forall y (\neg (y \in x))$

$x = \Lambda$

$\exists x (x = \Lambda)$

# Unchanged Standard Axioms

- Extensionality: $\forall x \forall y(\forall z(z \in x \Leftrightarrow z \in y \Rightarrow x = y))$
- Pairing: $\forall x \forall y \exists z(\forall t(t \in z \Leftrightarrow (t = x \vee t = y)))$
- Union: $\forall x \exists y \forall u(u \in y \Rightarrow \exists z(z \in x \wedge u \in z))$
- Power Set: $\forall x \exists y(\forall z(\forall t(t \in z \Rightarrow t \in x) \Rightarrow z \in y))$

# Modified Standard Axioms

- Separation Scheme: $\forall p_1,\ldots,p_n \forall x \exists y \forall z(z \in y \Leftrightarrow (z \in x \wedge (\Phi(z, p_1,\ldots,p_n) \vee z = \Lambda)))$
- Replacement Scheme: $\forall p_1,\ldots,p_n (\forall x \forall y \forall z((\Phi(x, y, p_1,\ldots,p_n) \wedge \Phi(x, z, p_1,\ldots,p_n)) \Rightarrow y = z) \Rightarrow \forall x \exists y \forall z(z \in y \Leftrightarrow \exists u(u \in x \wedge (\Phi(u, z, p_1,\ldots,p_n)) \vee z = \Lambda)))$
- Regularity: $\forall x(\exists y(y \neq \Lambda \wedge y \in x \Rightarrow \exists y(y \neq \Lambda \wedge y \in x \wedge \neg \exists z(z \neq \Lambda \wedge z \in y \wedge z \in x)))$

# Standard  Axiom of Infinite

$$\exists x(\varnothing \in x \wedge \forall y(y \in x \Rightarrow y \cup \{y\} \in x))$$

# Lambda Axiom of Infinite

$$\exists x \forall y (y \in x \Rightarrow y \cup \{y\} \in x)$$

# Empty Set Theorem

We build the singleton of Lambda by means of the axiom of the parts:

$$\wp(\Lambda) = \{\Lambda\}$$

- No set is included in Lambda. So Lambda has no parts. And the set of its parts is free of elements.
- But in the same way as an element belongs to the set of its parts, the pre-element Lambda belongs to the set of its parts.

# Empty Set by the Axiom of Pairing

With x = y = $\Lambda$, we have z = $\{\Lambda\}$

# Lambda and Singleton of Lambda

- $\exists x \forall y (\neg (y \in x))$
- $\exists x \forall y (y \in x \Rightarrow y = \Lambda)$
- $x = \Lambda$
- $x = \{\Lambda\}$

# Lambda and Contradictory Property

- What about the standard definition of empty set by means of a contradictory property?

$$\{x : x \neq x\}$$

- By the axiom of pre-element, $\Lambda$ must belong to this set
- $\{x : x \neq x\} = \{\Lambda\}$
- $(x = x \Leftrightarrow x \neq x) \Rightarrow x$ is a pre-element

# Singleton of Lambda and Standard Empty Set

We want to check the following equivalences:

- Do contain nothing $\equiv$ do not contain anything
- Free of sets ($\{\Lambda\}$) $\equiv$ free of sets and Lambda ($\emptyset$)

**Behaviour of Singleton of Lambda and of Standard Empty Set**

With x and y $\neq \{\Lambda\}$ and no common element

## Singleton of Lambda

- $x \cap y = \{\Lambda\}$
- $x \cap \{\Lambda\} = \{\Lambda\}$
- $x \cup \{\Lambda\} = x$
- $x \setminus \{\Lambda\} = x$
- $x \setminus x = \{\Lambda\}$
- $\{\Lambda\} \cap \{\Lambda\} = \{\Lambda\}$
- $\{\Lambda\} \cup \{\Lambda\} = \{\Lambda\}$
- $\{\Lambda\} \setminus \{\Lambda\} = \{\Lambda\}$

## Standard Empty Set

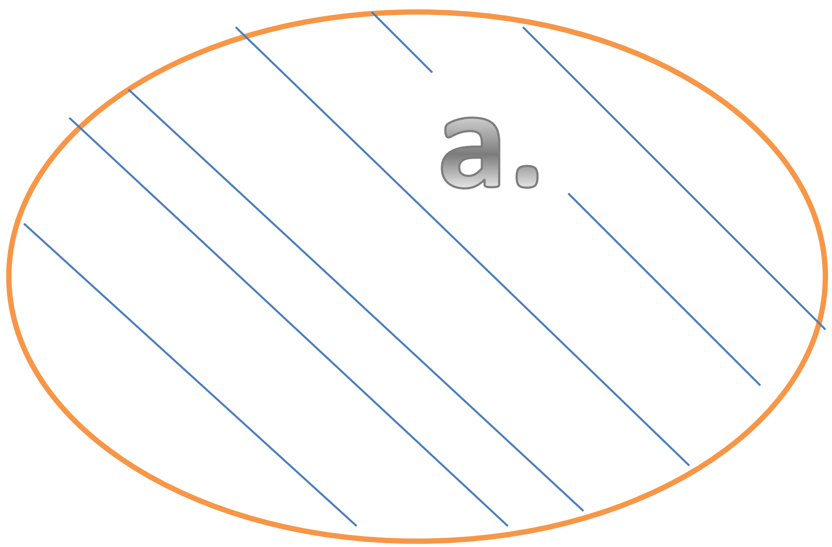
- $x \cap y = \varnothing$
- $x \cap \varnothing = \varnothing$
- $x \cup \varnothing = x$
- $x \setminus \varnothing = x$
- $x \setminus x = \varnothing$
- $\varnothing \cap \varnothing = \varnothing$
- $\varnothing \cup \varnothing = \varnothing$
- $\varnothing \setminus \varnothing = \varnothing$

# Conclusion

- Either the singleton of Lambda is the revised version of the standard empty set, or Set theory does not need empty set.

# Representation of Lambda

- a is an element
- Lambda must be conceived and seen as the zone around the element(s).



# Behaviour of Lambda

- Lambda is the non numerical form of zero, or zero is the numerical form of Lambda, i.e. of the absence.
- $x \cap y = \{\Lambda\}$
- $x \cap \Lambda = \{\Lambda\}$ ???
- $x \cup \Lambda = x$
- $x \setminus \Lambda = x$
- $\Lambda \cap \Lambda = \{\Lambda\}$ ???
- $\Lambda \cup \Lambda = \{\Lambda\}$ ???
- $\Lambda \setminus \Lambda = \{\Lambda\}$ ???

# Technical Interest of Lambda

- New definition of Set
- New definition of Empty Set
- No need of a contradictory property
- Uselessness of Empty Set Axiom and of Axiom of Existence of Set
- Simplification  of the Axiom of Infinite
- Distinction empty set/ur-element

# Conceptual Interest of Lambda

- Solution to the Puzzle of the Null Class (Russell)
- Distinction Empty set/void/zero (Lambda is not the empty set because it is not a set)

# Future Developments

- Logic and theory of Potential
- Solution to empty family intersection anomaly

# Questions

- L’axiome d’existence de Lambda que j’ai ajouté vous semble-t-il nécessaire?
- Que pensez-vous de l’appartenance de Lambda à l’ensemble construit au moyen de la définition contradictoire? Vous pose-t-elle un problème?
- Que pensez-vous du fait qu’on ne puisse soustraire Lambda à un ensemble dans une différence symétrique? Cela vous pose-t-il un problème?

- Que pensez-vous de la construction du Singleton de Lambda au moyen de l’axiome de l’ensemble des parties?
- Que pensez-vous de l’étude du comportement de Lambda dans des opérations comme l’intersection, l’union…?