Lambda Theory Model

Co-consistency of Lambda theory with ZF(C) theory

laurent.dubois@ulb.ac.be

Language

- $\in$, =, Λ (constant)
- First-order language

Abbreviations

- $\forall^* x \equiv \forall x \neq \Lambda$
- $\exists^* x \equiv \exists x \neq \Lambda$
- Λ-set $\equiv$ set that contains Λ

Theory

- $\forall x(x \neq \Lambda \Rightarrow \Lambda \in x)$ (1)
- $\forall x((x = \Lambda) \vee (\Lambda \in x))$ (2) (contraposition of 1)
- $\neg\exists x(x \neq \Lambda \wedge \neg(\Lambda \in x))$ (3) (dual of 2)
- $\forall x(\neg(x \in \Lambda))$ (4)
- $\forall x(x \neq \Lambda \Leftrightarrow \Lambda \in x)$ (5) (1 + 4)

σ-axioms

- Extensionality: $\forall x \forall y(\forall z((z \in x \Leftrightarrow z \in y) \Rightarrow x = y))$

- Triple: $\forall x \forall y \exists z(\forall t(t \in z \Leftrightarrow t = x \vee t = y \vee t = \Lambda)$.

Or $\forall x \forall y \exists z(x \in z \wedge y \in z)$ + separation.

- Union: $\forall x \exists y \forall u(u \in y \Rightarrow \exists z(z \in x \wedge u \in z))$

- Power Set: $\forall x \exists y(\forall z((\forall t(t \in z \Rightarrow t \in x) \wedge \Lambda \in z) \Rightarrow z \in y) \wedge \Lambda \in y)$

σ-axioms

- Separation Scheme: $\forall p_1,\ldots,p_n \forall x \exists y \forall z(z \in y \Leftrightarrow (z \in x \wedge (\Phi(z, p_1,\ldots,p_n) \vee z = \Lambda)))$
- Replacement Scheme: $\forall p_1,\ldots,p_n (\forall x \forall y \forall z((\Phi(x, y, p_1,\ldots,p_n) \wedge \Phi(x, z, p_1,\ldots,p_n)) \Rightarrow y = z) \Rightarrow \forall x \exists y \forall z(z \in y \Leftrightarrow \exists u(u \in x \wedge (\Phi(u, z, p_1,\ldots,p_n)) \vee z = \Lambda)))$
- Infinity: $\exists x \forall y(y \in x \Rightarrow y \cup \{y\} \in x)$
- Regularity: $\forall x(\exists y(y \neq \Lambda \wedge y \in x \Rightarrow \exists y(y \neq \Lambda \wedge y \in x \wedge \neg\exists z(z \neq \Lambda \wedge z \in y \wedge z \in x)))$

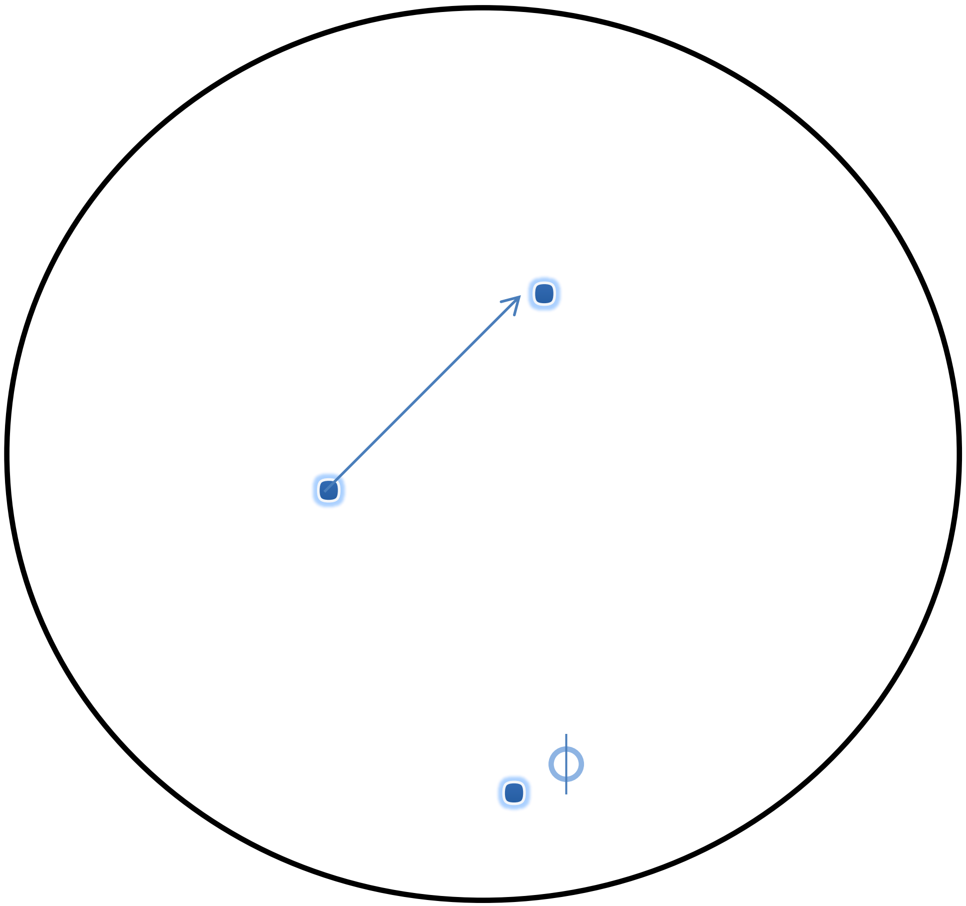
σ*-axioms

- Extensionality: $\forall^*x\forall^*y(\forall^*z((z \in x \Leftrightarrow z \in y) \Rightarrow x = y))$
- Triple: $\forall^*x\forall^*y\exists^*z(x \in z \wedge y \in z)$ + separation.
- Union: $\forall^*x\exists^*y\forall^*u(u \in y \Rightarrow \exists^*z(z \in x \wedge u \in z))$
- Power Set: $\forall^*x\exists^*y(\forall^*z((\forall^*t(t \in z \Rightarrow t \in x) \wedge \Lambda \in z) \Rightarrow z \in y) \wedge \Lambda \in y)$

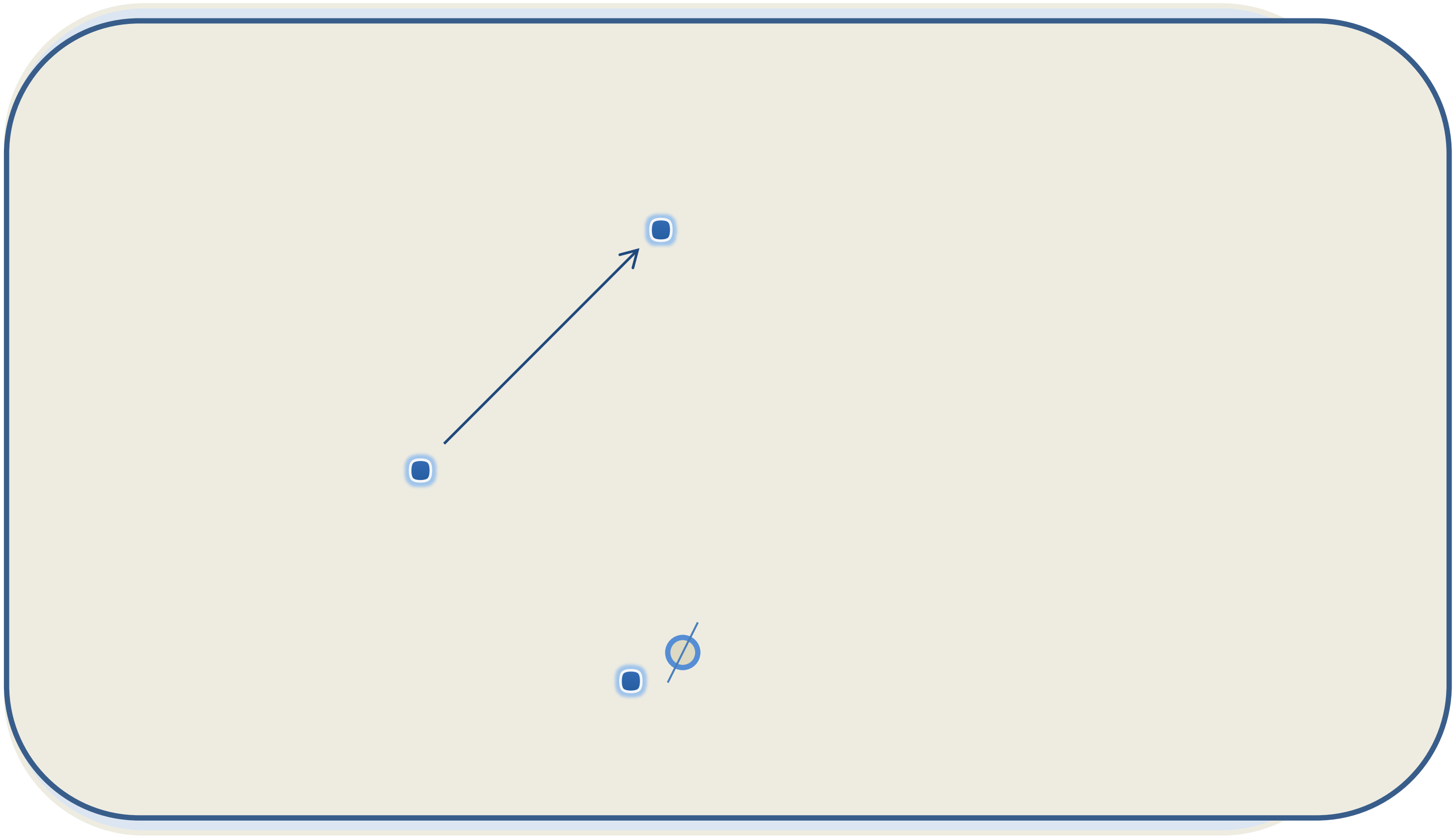
σ*-axioms

- Separation Scheme: $\forall^* p_1,\ldots,p_n \forall^* x \exists^* y \forall^* z(z \in y \Leftrightarrow (z \in x \wedge (\Phi(z, p_1,\ldots,p_n))))$
- Replacement Scheme: $\forall p_1,\ldots,p_n (\forall^* x \forall^* y \forall^* z((\Phi(x, y, p_1,\ldots,p_n) \wedge \Phi(x, z, p_1,\ldots,p_n)) \Rightarrow y = z) \Rightarrow \forall^* x \exists^* y \forall^* z(z \in y \Leftrightarrow \exists^* u(u \in x \wedge (\Phi(u, z, p_1,\ldots,p_n)))))$
- Infinity: $\exists^* x \forall^* y(y \in x \Rightarrow y \cup \{y\} \in x)$
- Regularity: $\forall^* x(\exists^* y(y \in x) \Rightarrow \exists^* y(y \in x \wedge \neg\exists^* z(z \in y \wedge z \in x)))$

ZF Universe: $\mathbb{U}, \in, =$

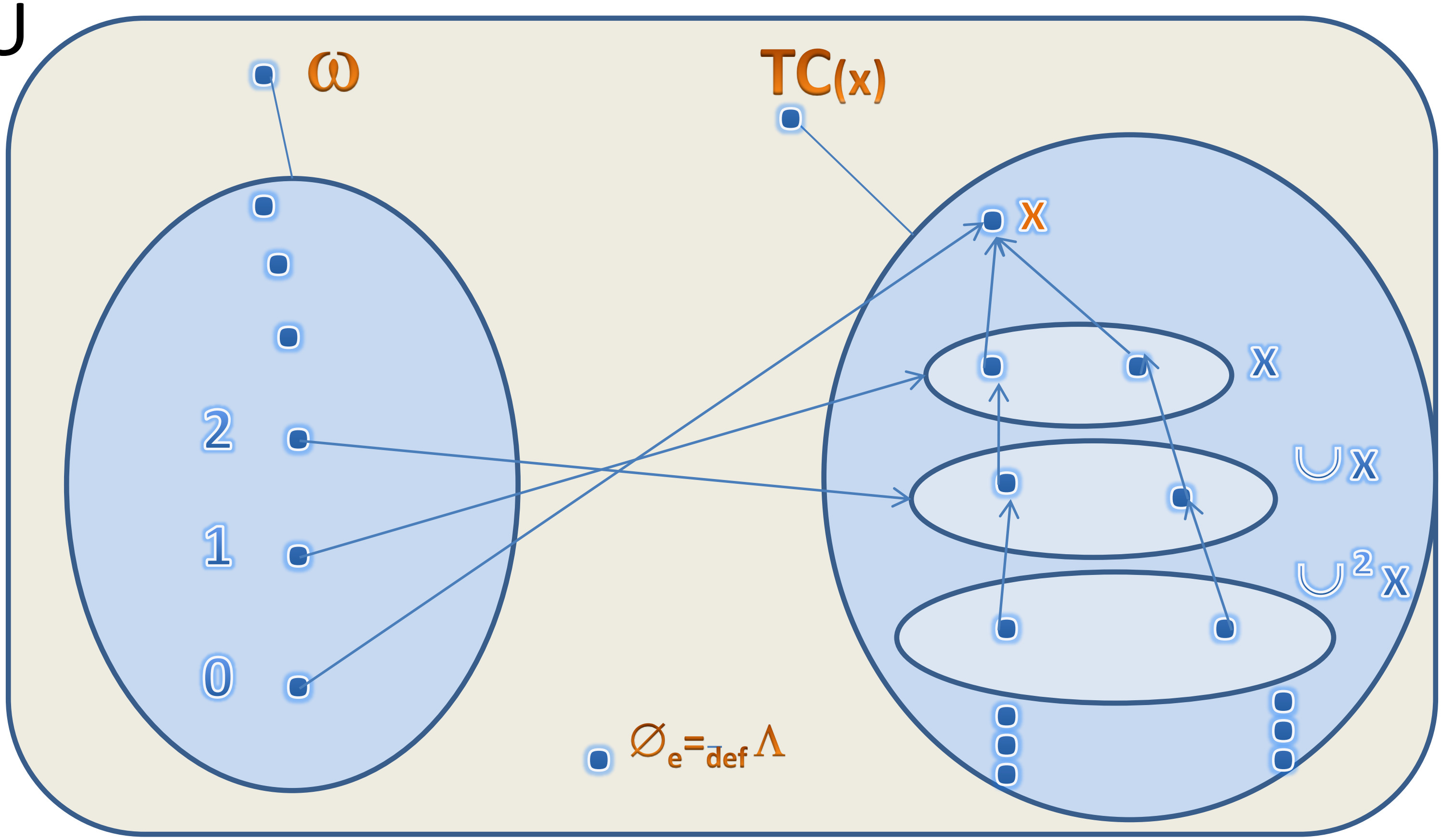


ZF Universe: $\mathbb{U}, \in, =$



Transitive Closure

- U

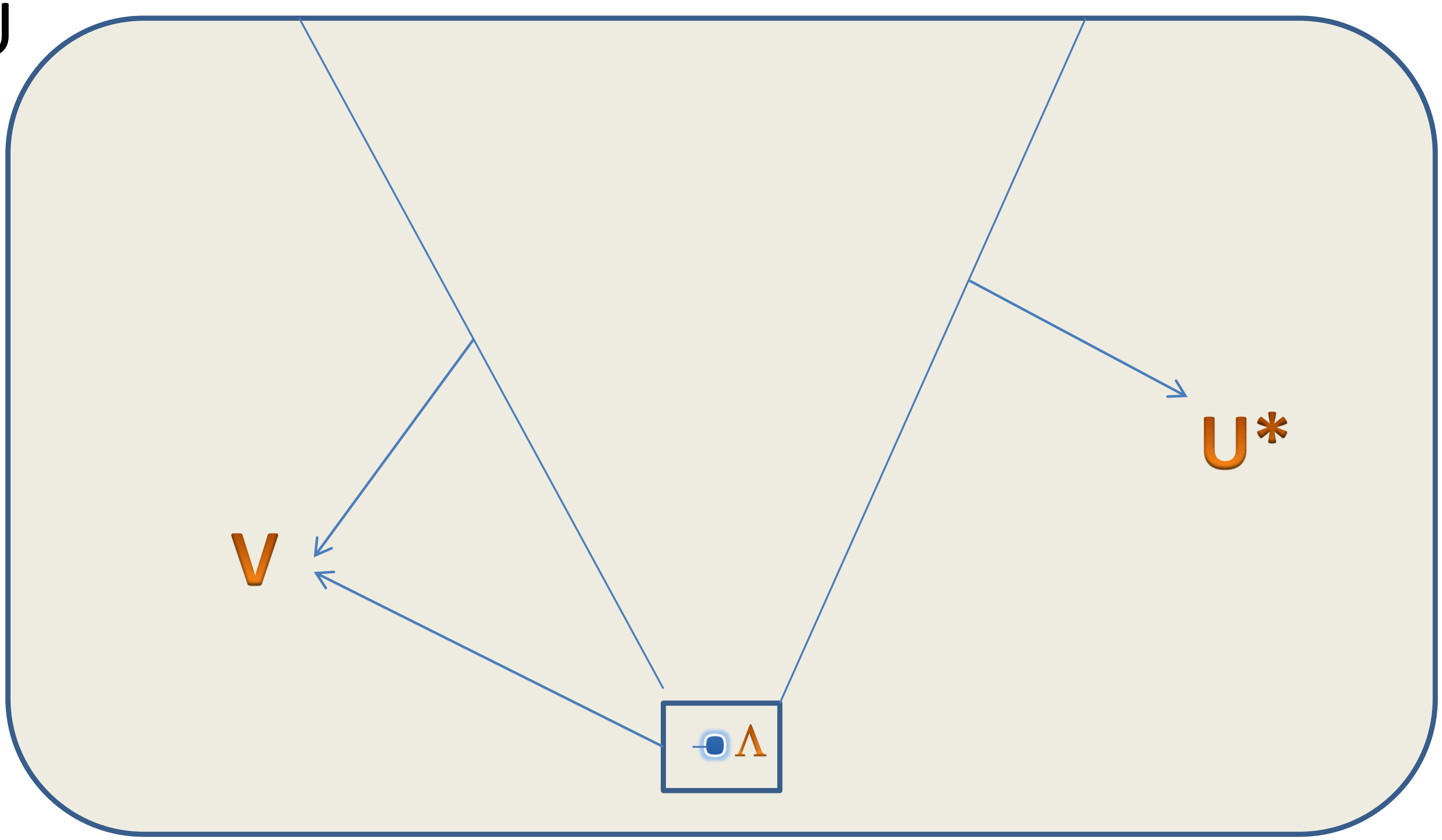


Transitive Closure

- $TC(x) =_{def.} \{x\} \cup [\bigcup\{x, \bigcup x, \bigcup^2 x, \bigcup^3 x, \ldots\}]$
- $\bigcup\{x, \bigcup x, \bigcup^2 x, \bigcup^3 x, \ldots\} = x \cup (\bigcup x) \cup (\bigcup^2 x) \ldots$
- $\{0, 1, 2, 3\ldots\} = \omega$
- We get the transitive closure by replacement scheme.

Lambda-Universe and ZF-Universe

- U



Terminology

- Λ is « the Nothing », « the Void ».

- $\mathbb{U}^*$ is the class of sets, i.e. the class of x such that $\forall z \in TC(x)(z \neq \Lambda \Rightarrow \Lambda \in z)$.

Lambda-Universe

- $\mathbb{V}$, $\in_{\wedge\mathbb{V}}$, $=_{\wedge\mathbb{V}}$

- $\mathbb{V} = \mathbb{U}^* \cup \{\Lambda\}$

- Trick: normal $\forall$, $\exists$: $\forall$x $\equiv$ $\forall$x of $\mathbb{V}$; $\exists$x $\equiv$ $\exists$x of $\mathbb{V}$
- : restricted $\forall$, $\exists$: $\forall$*x $\equiv$ $\forall$x of $\mathbb{U}^*$; $\exists$*x $\equiv$ $\exists$x of $\mathbb{U}^*$, i.e. $\forall$x, $\exists$x of $\mathbb{V}$ such that $x \neq \Lambda$.

Consequences (to be checked)

- Be σ a ZF axiom: σ^* is true ($\sigma^* \equiv$ all the quantifiers are *).
- $\forall^* x(\Lambda \in x)$
- $\exists z \in x \leftrightarrow x \neq \Lambda$

EXT* and EXT

- EXT* true. Proof:
- We must show that: $(\forall x \forall y(\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y))^* \equiv \forall^* x \forall^* y(\forall^* z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$
- Preliminary: $\mathbb{U}^*$ is transitive: $a \in b$ "$\in$" $\mathbb{U}^* \Rightarrow a$ "$\in$" $\mathbb{U}^*$.
- x, y "$\in$" $\mathbb{U}^* \Rightarrow x \neq \Lambda, y \neq \Lambda$
- Notation: $x \mathrel{\overset{=}{\wedge}} y$ means that x and y have same elements and are equal by the bottom.

- Phenomenon: if x, y $\in \mathbb{U}^*$ and (x $\doublebarwedge$ y)$_{\mathbb{U}^*}$, then (x $\doublebarwedge$ y) (in ZF universe), so x = y.

Terms

- Term: $\{x|\ \varphi(x,\ldots)\}$ in the language: $\in$, =, Λ.
- Don't forget we work in $\mathbb{V}$, $\in$, =.
- What about "$\{x|\ \varphi(x\ldots)\}$ "?
- $\{x|\ \varphi(x,\ldots)\}$ is a "a" such that: (hope) $\forall y(y \in a \Leftrightarrow \varphi(y\ldots))$ - in $\mathbb{V}$.
- Probably less effective a-terms.

Terms

- $\{x \mid \varphi(x,\ldots)\}^*$ is a ''b'' such that: (hope) $\forall^* y(y \in b \Leftrightarrow \varphi^*(y\ldots))$ - in $\mathbb{U}^*$.
- To be checked (very probable): (ZF)*, all the standard operations of ZF.

Example of term

- $\{\Lambda\} = \varnothing_{ens.}$ (in $\mathbb{U}^*$)
- $\{\Lambda\} =_{def.} \{x | x = \Lambda\}$
- Two ways to get $\{\Lambda\}$:

(1) pairing + separation:

- pairing: with $x = y = \Lambda$, we know that $\Lambda \in z$.
- Separation: we keep from z the x such that $x = \Lambda$ and we have the singleton of Lambda.

$\{\Lambda\}$

(2) axiom of the parts applied to Λ:

$\exists y \forall z(\forall t(t \in z \Rightarrow t \in \Lambda) \Rightarrow z \in y)$

As nothing is in Λ, nothing can be in z, so z has to be « nothing » (Λ), and y must contain Λ: y = $\wp(\Lambda) = \{\Lambda\}$.

$\wp(\Lambda)$ is the only set containing only « nothing », that is $\wp(\Lambda) = \varnothing_{ens.}$.

Example of checking of ZF axiom in $\mathbb{U}^*$

- Axiom of pairing: $(\forall x \forall y \exists z \forall t(t \in z \Leftrightarrow (t = x \vee t = y)))^* \equiv \forall^* x \forall^* y \exists^* z \forall^* t(t \in z \Leftrightarrow (t = x \vee t = y))$
- Thanks to separation, we are sure that the set contains Λ; so « pairing » must become « triple ».
- The true triple in $\mathbb{U}^*$: $\{x, y, \Lambda\}$
- We are sure that $\{x, y, \Lambda\}$ is in $\mathbb{U}^*$ thanks to TC.
- What is $\{x, y, \Lambda\}$? It is the standard pair with Λ in addition.

- $\forall x \forall y \exists z \forall t (t \in z \Leftrightarrow (t = x \vee t = y \vee t = \Lambda))$

Set of the parts

- $\wp$ *a $=_{\text{def.}}$ $\{b \cup \{\Lambda\} | b \text{ in } \mathbb{U}\} \cup \{\Lambda\}$
- General rule (to be checked) for $\{x|\varphi(x,\ldots)\}^*$ (this element is always in $\mathbb{U}^*$, so different from Λ): $\{x \cup \{\Lambda\}|\varphi(x,\ldots)\} \cup \{\Lambda\}$ and x in $\mathbb{U}^*$.
- A bit harder: what about $\wp$a? That means set of the parts in $\mathbb{V}$ rather than in $\mathbb{U}^*$.
- Preliminary: ''$x \subset y$'': $x \subset y \Leftrightarrow \forall t((t \in x \Rightarrow t \in y) \wedge \Lambda \in x)$

Set of the parts

- We have seen that $\wp(\Lambda) = \{\Lambda\}$.
- From there, we can use the general rule $\{x \cup \{\Lambda\} | \varphi(x,\ldots)\} \cup \{\Lambda\}$ as in $\mathbb{U}^*$ to get all other sets of the parts.

Current operations

- I$\cap$ tersection and $\cup$nion are wrong in $\mathbb{U}^*$ and trivially true in $\mathbb{V}$:
- $z = (x \cap y) \Leftrightarrow \forall t(t \in z \Leftrightarrow (t \in x \wedge t \in y))$
- $z = (x \cap y)^* \Leftrightarrow z = x \cap^* y$
- $(\forall t(t \in z \Leftrightarrow (t \in x \wedge t \in y)))^* \Leftrightarrow \forall^* t(t \in z \Leftrightarrow (t \in x \wedge t \in y))$
- $z = (x \cup y) \Leftrightarrow \forall t(t \in z \Leftrightarrow (t \in x \vee t \in y))$
- $z = (x \cup y)^* \Leftrightarrow z = x \cup^* y$
- $(\forall t(t \in z \Leftrightarrow (t \in x \vee t \in y)))^* \Leftrightarrow \forall^* t(t \in z \Leftrightarrow (t \in x \vee t \in y))$
- It can not function in $\mathbb{U}^*$ since it excludes Λ of z.
- A solution: $z = \{t | \varphi(t,\ldots)\} \cup \{\Lambda\}$ and t in $\mathbb{U}^*$. Ad hoc?
- No problem in $\mathbb{V}$: Λ is taken into account.

- $\Gamma_\Lambda \models \cup \Rightarrow \Gamma_\Lambda \models \cap$???
- $\Gamma_\Lambda{}^* \neg\models \cap \Rightarrow \Gamma_\Lambda{}^* \neg\models \cup$
- Ma notation est-elle légale? Mes raisonnements corrects?

- Γ_Λ for Lambda theory
- $\Gamma_\Lambda{}^*$ for Lambda theory with $\forall^*$, $\exists^*$
- Γ_{ZF} for ZF theory

Empty family intersection and union

- Particular cases: empty family intersection and union.
- What does « empty family » mean?
- Classicaly: $\bigcup_{x\in\{\varnothing\}}X = \varnothing$; $\bigcup_{x\in\{\}}X = \varnothing$, $\bigcap_{x\in\{\varnothing\}}X = \varnothing$; $\bigcap_{x\in\{\}}X = \mathbb{U}$.
- In Lambda theory:
- $\bigcup_{x\in\{\{\Lambda\}\}}X \equiv \bigcup\{\Lambda\}$ and $\bigcup\{\Lambda\} = \{\Lambda\}$; $\bigcup_{x\in\{\Lambda\}}X \equiv \bigcup\Lambda$ and $\bigcup\Lambda = \Lambda$
- $\bigcap_{x\in\{\{\Lambda\}\}}X \equiv \bigcap\{\Lambda\}$ and $\bigcap\{\Lambda\} = \{\Lambda\}$; $\bigcap_{x\in\{\Lambda\}}X \equiv \bigcap\Lambda$ and $\bigcap\Lambda = \Lambda$

Empty family intersection and union

- In $\bigcap_{x \in \{\}} X$, if it is not true that $x \in X$ for each X of $\varnothing$, then it must exist a X such that $x \notin X$; since there is no X in $\varnothing$, no X puts at fault the condition, so any X satisfies it, and the x's specified by the condition exhaust the universe $\mathbb{U}$.
- Formally: $\forall X(x \in X) \equiv \neg\exists X \neg(x \in X)$
- The right side is read before the left side for the interpretation.

Empty family intersection and union

- For $\bigcup_{x\in\{\}}X$: $\exists X(x \in X) \equiv \neg\forall X\neg(x \in X)$
- Left side is read first: no X, so $\exists X$ is false.
- Right side: no X, so $\forall X(x \in X)$ is false. So we have at least $\neg\forall X(x \in X)$. At the extreme, we have $\forall X\neg(x \in X)$. There is no X such that $x \in X$, so no X to contradict $\forall X\neg(x \in X)$. So, $\neg\forall X\neg(x \in X)$ is invalidated and $\bigcup_{x\in\{\}}X = \varnothing$.
- Anomalies:

- Condition on $\bigcap$ stronger than that on $\bigcup$. If $\neg\exists X\neg(x \in X)$ is true, $\neg\forall X\neg(x \in X)$ is true. If $\bigcap_{x\in\{\}}X$ gives $\mathbb{U}$, $\bigcup_{x\in\{\}}X$ must give $\mathbb{U}$ too.
- $\bigcap \subset \bigcup$: if $\bigcup_{x\in\{\}}X = \varnothing$, $\bigcap_{x\in\{\}}X = \varnothing$.

- Solution: in $\mathbb{V}$, $\exists X=\Lambda(\neg(x \in X))$, so $\bigcap_{x\in\{\Lambda\}}X = \Lambda$. And $\bigcup_{x\in\{\Lambda\}}X = \Lambda$.

Symmetric d\fference

- Symmetric d\fference needs slight change:
- $(z = x \setminus y \Leftrightarrow \forall t(t \in z \Leftrightarrow ((t \in x) \wedge ((t \notin y) \vee t = \Lambda))))^*$
- $(\forall t(t \in z \Leftrightarrow ((t \in x) \wedge ((t \notin y) \vee t = \Lambda))))^* \Leftrightarrow \forall^* t(t \in z \Leftrightarrow ((t \in x) \wedge ((t \notin y) \vee t = \Lambda)))$

- So does in$\subset$lusion:
- $x \subset y \Leftrightarrow \forall t((t \in x \Rightarrow t \in y) \wedge \Lambda \in x)$

Singleton of Lambda and Standard Empty Set

We want to check the following equivalences for sets:

- Do contain nothing $\equiv$ do not contain anything
- Free of sets ($\{\Lambda\}$) $\equiv$ free of sets and Lambda ($\varnothing_{ZF}$)
- How can we proceed? Sets in $\mathbb{U}^*$ are standard (ZF) sets to which Λ has been added. So, $\{\Lambda\}$ is exactly in the same relation with Λ-sets as $\varnothing_{ZF}$ with standard sets.

Singleton of Lambda, Lambda and Contradictory Property

- What about the standard definition of empty set by means of a contradictory property?

$$\{x : x \neq x\}$$

- Thanks to the disjunction added to the comprehension axiom scheme, for any set x, there is a set y that contains at least Lambda, without the necessity for Lambda to satisfy any property.
- So, $\{x : x \neq x\}$ is not a set and can define Lambda: $\{x : x \neq x\} = \Lambda$.

A definition of Lambda

- In V, $\exists x \forall y (\neg (y \in x))$: $x = \Lambda$
  $\exists x (x = \Lambda)$

- $\exists x \forall y (y \in x \Rightarrow y = \Lambda)$: $x = \{\Lambda\}$

Behaviour of Lambda and of Standard Empty Set

Lambda

- $x \cap y = \{\Lambda\}$
- $x \cap \Lambda = \Lambda$
- $x \cup \Lambda = x$
- $x \setminus \Lambda = x$
- $x \setminus x = \{\Lambda\}$
- $\Lambda \cap \Lambda = \Lambda$
- $\Lambda \cup \Lambda = \Lambda$
- $\Lambda \setminus \Lambda = \Lambda$

Standard Empty Set

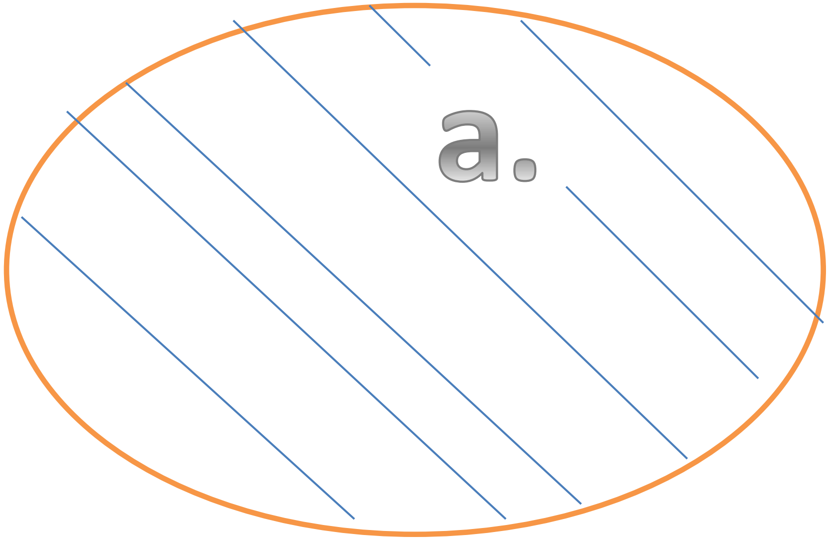
- $x \cap y = \varnothing$
- $x \cap \varnothing = \varnothing$
- $x \cup \varnothing = x$
- $x \setminus \varnothing = x$
- $x \setminus x = \varnothing$
- $\varnothing \cap \varnothing = \varnothing$
- $\varnothing \cup \varnothing = \varnothing$
- $\varnothing \setminus \varnothing = \varnothing$

Why Lambda can not be assimilated to Classical Empty Set

- Product and difference of x and y give $\{\Lambda\}$ and not Λ.
- Lambda belongs to any set while Empty set and singleton of Lambda don’t.
- Lambda is not a set.
- Contradictory property does not help to define a set since $\{x : x \neq x\}$ does not contain Λ.
- $\wp(\Lambda)$ gives $\varnothing$ (no set included in Λ) and we add Λ to $\varnothing$ by the axiom of pre-element.

Representation of Lambda

- a is an element
- Lambda must be conceived and seen as the free zone around the element(s).



Definition of Element, Pre-element and Set

- x is an element $\Leftrightarrow$ x is a set
- x is a pre-element $\Leftrightarrow$ $\forall$y(y is an element $\Rightarrow$ x $\in$ y)
- x is a set $\Leftrightarrow$ $\exists$y(y $\in$ x)