Temps avant chute d'un objet leger sur un attracteur gravitationnel.

$$\boxed{\frac{d^2r}{dt^2} = -\frac{GM}{r^2}} \qquad \text{or :} \quad \frac{dr}{dt} = v \Leftrightarrow \frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt} = \frac{dv}{dr}\cdot v$$

$$\frac{d^2r}{dt^2} = \frac{dv}{dr}\cdot v \quad \Leftrightarrow \quad \frac{dv}{dr}\cdot v = -\frac{GM}{r^2} \Leftrightarrow v\,dv = -GM\frac{dr}{r^2}$$

Aucune vitesse initiale : $v(r)\Big|_{r=r_0} = 0$

$$\int_0^{v(r)} v\,dv = GM\int_{r_0}^{r} -\frac{dr}{r^2} \Leftrightarrow \left[\frac{v^2}{2}\right]_0^{v(r)} = GM\left[\frac{1}{r}\right]_{r_0}^{r} \Leftrightarrow \frac{v(r)^2}{2} = GM\left(\frac{1}{r}-\frac{1}{r_0}\right)$$

$$\Leftrightarrow \quad v(r)^2 = 2GM\left(\frac{1}{r}-\frac{1}{r_0}\right) \Leftrightarrow v(r) = \pm\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)}$$

L'objet se rapproche du pole soit $v(r) < 0$

$$v(r) = -\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)} \qquad \text{or :} \quad v(r) = \frac{dr}{dt}$$

$$\frac{dr}{dt} = -\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)} \Leftrightarrow \frac{dr}{\sqrt{\frac{1}{r}-\frac{1}{r_0}}} = -\sqrt{2GM}\,.dt$$

$$-\sqrt{2GM}\int_0^t dt = \int_{r_0}^{0}\frac{dr}{\sqrt{\frac{1}{r}-\frac{1}{r_0}}} \Leftrightarrow \sqrt{2GM}\,.\,t = -\int_{r_0}^{0}\frac{dr}{\sqrt{\frac{1}{r}-\frac{1}{r_0}}}$$

$$\sqrt{2GM}\,.\,t = -\int_{r_0}^{0}\sqrt{\frac{\frac{1}{r_0}}{\frac{1}{r_0}-\frac{1}{r_0}}}\sqrt{r_0}\,.dr \Leftrightarrow \frac{\sqrt{2GM}}{\sqrt{r_0}}\,t = -\int_{r_0}^{0}\sqrt{\frac{\frac{r}{r_0}}{1-\frac{r}{r_0}}}\,dr$$

$$\sqrt{\frac{2GM}{r_0}}\,.\,t = -\int_{r_0}^{0}\sqrt{\frac{\frac{r}{r_0}}{1-\frac{r}{r_0}}}\,dr$$

posons : $\sin^2(u) = \frac{r}{r_0} \Leftrightarrow u = \operatorname{Arcsin}\sqrt{\frac{r}{r_0}}$

$$\frac{dr}{du} = r_0\cdot\frac{d}{du}\left[\sin^2(u)\right] = r_0\cdot\frac{d}{du}\left(\frac{1-\cos(2u)}{2}\right) = \frac{r_0}{2}\frac{d}{du}\left[-\cos(2u)\right]$$

$$= \frac{r_0}{2}\left[2\sin(2u)\right] = r_0\,.\sin(2u) = 2r_0\,.\sin(u).\cos(u)$$

$dr = 2r_0\sin(u)\cos(u)du.$

$$\begin{cases} u(r_0) = \operatorname{Arcsin}(1) = \frac{\pi}{2} \\ u(0) = \operatorname{Arcsin}(0) = 0 \end{cases}$$

$$\sqrt{\frac{2GM}{r_0}}\,.\,t = \int_{u(0)}^{u(r_0)}\sqrt{\frac{\sin^2(u)}{1-\sin^2(u)}}\,.\,2.r_0.\sin(u).\cos(u)\,du$$

$$= \int_0^{\pi/2}\frac{2\sin(u)\,r_0\sin(u)\cos(u)}{\cos(u)}du = 2r_0\int_0^{\pi/2}\sin^2(u)du$$

$$= 2r_0\int_0^{\pi/2}\left(\frac{1-\cos(2u)}{2}\right)du = 2r_0\left[\int_0^{\pi/2}\frac{1}{2}du - \int_0^{\pi/2}\frac{\cos(2u)}{2}du\right]$$

$$= 2r_0\left[\frac{\pi}{4} - \underbrace{\left[\frac{\sin(2u)}{4}\right]_0^{\pi/2}}_{=0}\right] = \frac{\pi r_0}{2}$$

$$\Leftrightarrow \quad \sqrt{\frac{2GM}{r_0}}\,.\,t = \frac{\pi r_0}{2} \Leftrightarrow \boxed{t = \pi\sqrt{\frac{r_0^3}{8GM}}}$$