INTRODUCTION TO DIFFERENTIAL TOPOLOGY

Joel W. Robbin
UW Madison

Dietmar A. Salamon
ETH Zürich

9 June 2011

Preface

These are notes for the lecture course *"Differential Geometry II"* held by the second author at ETH Zürich in the spring semester 2011. A prerequisite is the foundational chapter about smooth manifolds in [16] as well as some basic results about geodesics. For the benefit of the reader we summarize the relevant material in Chapter 1 of the present manuscript.

The first half of the book deals with degree theory, the Pontryagin construction, intersection theory, and Lefschetz numbers. In this part we follow closely the exposition of Milnor in [10]. For the additional material on intersection theory and Lefschetz numbers an excellent reference is the book by Guillemin and Pollak [5].

The second half of the book is devoted to differential forms and deRham cohomology. It begins with an elemtary introduction into the subject and continues with some deeper results such as Poincaré duality, the Čech-deRham complex, and the Thom isomorphism theorem. Many of our proofs in this part are taken from the classical textbook of Bott and Tu [2] which is also a highly recommended reference for a deeper study of the subject (including sheaf theory, homotopy theory, and characteristic classes).

2 May 2011 *Joel W. Robbin and Dietmar A. Salamon*

Contents

Chapter 1

Foundations

Chapter 2

Degree Theory

Chapter 3

The Poincaré–Hopf Theorem

Chapter 4

The Pontryagin Construction

Chapter 5

Intersection Theory

Chapter 6

Lefschetz Numbers

Chapter 7

Differential Forms

This chapter begins with an elemntary discussion of differential forms on manifolds. Section 7.1 explains the exterior algebra of a real vector space and its relation to the determinant of a square matrix. In Section 7.2 we introduce differential forms on manifolds, their exterior products and pull-backs, and the exterior differential in local coordinates as well as globally. The section closes with a brief discussion of deRham cohomology. Section 7.3 introduces the integral of a compactly supported differential form of top degree over an oriented manifold and and contains a proof of Stokes' theorem. In Section 7.4 we prove Cartan's formula for the Lie derivative of a differential form in the direction of a vector field. We use it in Section 7.5 to show that a top degree form on a compact connected oriented smooth manifold without boundary is exact if and only if its integral vanishes. As applications of these results we prove the degree theorem and the Gauss–Bonnet formula. The chapter closes with an introduction to Moser isotopy for volume forms.

7.1 Exterior Algebra

7.1.1 Alternating Forms

We assume throughout that V is an m-dimensional real vector space and $k \in \mathbb{N}$ is a positive integer. Let S_k denote the permutation group on k elements, i.e. the group of all bijective maps $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$. The group operation is given by composition. There is a group homomorphism $\varepsilon : S_k \to \{\pm 1\}$ defined by

$$\varepsilon(\sigma) := (-1)^{\nu}, \qquad \nu(\sigma) := \# \left\{ (i,j) \in \{1, \dots, k\}^2 \,|\, i < j,\, \sigma(i) > \sigma(j) \right\}.$$

Definition 7.1. *An* **alternating** k**-form** *on* V *is a multi-linear map*

$$\omega : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

satisfying

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \varepsilon(\sigma)\omega(v_1, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$ *and all* $\sigma \in S_k$*. An* **alternating** 0**-form** *is by definition a real number. The vector space of all alternating* k*-forms on* V *will be denoted by*

$$\Lambda^k V^* := \left\{ \omega : V^k \to \mathbb{R} \,|\, \omega \textit{ is an alternating } k\textit{-form} \right\}.$$

For $\omega \in \Lambda^k V^*$ *the integer* $k =: \deg(\omega)$ *is called the* **degree** *of* ω*.*

Example 7.2. The space of alternating 0-forms is the real line: $\Lambda^0 V^* = \mathbb{R}$.

Example 7.3. The space of alternating 1-forms is the dual space of V:

$$\Lambda^1 V^* = V^* := \mathrm{Hom}(V, \mathbb{R}).$$

In the case $V = \mathbb{R}^m$ denote by $dx^i : \mathbb{R}^m \to \mathbb{R}$ the projection onto the ith coordinate, i.e.

$$dx^i(\xi) := \xi^i$$

for $\xi = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$ and $i = 1, \ldots, m$. Then the linear functionals $dx^1, \ldots, dx^m$ form a basis of the dual space $(\mathbb{R}^m)^* = \Lambda^1(\mathbb{R}^m)^*$.

Example 7.4. An alternating 2-form on V is a skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$ so that

$$\omega(v, w) = -\omega(w, v)$$

for all $v, w \in V$. In the case $V = \mathbb{R}^m$ an alternating 2-form can be written in the form

$$\omega(\xi, \eta) = \langle \xi, A\eta \rangle$$

for $\xi, \eta \in \mathbb{R}^m$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^m$ and $A = -A^T \in \mathbb{R}^{m \times m}$ is a skew-symmetric matrix. Thus

$$\dim \Lambda^2 V^* = \frac{m(m-1)}{2}.$$

for every m-dimensional real vector space V.

Definition 7.5. *Let $\mathcal{I}_k = \mathcal{I}_k(m)$ denote the set of ordered k-tuples*

$$I = (i_1, \dots, i_k) \in \mathbb{N}^k, \qquad 1 \le i_1 < i_2 < \cdots < i_k \le m.$$

For $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ the alternating k-form

$$dx^I : \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R}$$

is defined by

$$dx^I(\xi_1, \dots, \xi_k) := \det \begin{pmatrix} \xi_1^{i_1} & \xi_2^{i_1} & \cdots & \xi_k^{i_1} \\ \xi_1^{i_2} & \xi_2^{i_2} & \cdots & \xi_k^{i_2} \\ \vdots & \vdots & & \vdots \\ \xi_1^{i_k} & \xi_2^{i_k} & \cdots & \xi_k^{i_k} \end{pmatrix} \tag{7.1}$$

for $\xi_j = (\xi_j^1, \dots, \xi_j^m) \in \mathbb{R}^m$, $j = 1, \dots, k$.

Lemma 7.6. *The elements dx^I for $I \in \mathcal{I}_k$ form a basis of $\Lambda^k(\mathbb{R}^m)^*$. Thus, for every m-dimensional real vector space V, we have*

$$\dim \Lambda^k V^* = \binom{m}{k}, \qquad k = 0, 1, \dots, m,$$

and $\Lambda^k V^* = 0$ for $k > m$.

Proof. The proof relies on the following three observations.

(1) Let $e_1, \dots, e_m$ denote the standard basis of $\mathbb{R}^m$ and fix an element $J = (j_1, \dots, j_k) \in \mathcal{I}_k$. Then, for every $I \in \mathcal{I}_k$, we have

$$dx^I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \ne J. \end{cases}$$

(2) For every $\omega \in \Lambda^k(\mathbb{R}^m)^*$ we have

$$\omega = 0 \qquad \Longleftrightarrow \qquad \omega(e_{i_1}, \dots, e_{i_k}) = 0 \quad \forall I = (i_1, \dots, i_k) \in \mathcal{I}_k.$$

(3) Every $\omega \in \Lambda^k(\mathbb{R}^m)^*$ can be written as

$$\omega = \sum_{I \in \mathcal{I}_k} \omega_I dx^I, \qquad \omega_I := \omega(e_{i_1}, \dots, e_{i_k}).$$

Here assertions (1) and (2) follow directly from the definitions and assertion (3) follows from (1) and (2). That the dx^I span the space $\Lambda^k(\mathbb{R}^m)^*$ follows immediately from (3). We prove that the dx^I are linearly independent: Let $\omega_I \in \mathbb{R}$ for $I \in \mathcal{I}_k$ be a collection of real numbers such that $\omega := \sum_I \omega_I dx^I = 0$; then, by (1), we have $\omega(e_{j_1}, \dots, e_{j_k}) = \omega_J$ for $J = (j_1, \dots, j_k) \in \mathcal{I}_k$ and so $\omega_J = 0$ for every $J \in \mathcal{I}_k$. This proves the lemma. □

7.1.2 Exterior Product

Let $k, \ell \in \mathbb{N}$ be positive integers. The set $S_{k,\ell} \subset S_{k+\ell}$ of (k,ℓ)-**shuffles** is the set of all permutations in $S_{k+\ell}$ that leave the order of the first k and of the last ℓ elements unchanged:

$$S_{k,\ell} := \{\sigma \in S_{k+\ell} \,|\, \sigma(1) < \cdots < \sigma(k),\, \sigma(k+1) < \cdots < \sigma(k+\ell)\}\,.$$

The terminology arises from *shuffling a card deck* with $k+\ell$ cards.

Definition 7.7. *The* **exterior product** *of two alternating forms $\omega \in \Lambda^k V^*$ and $\tau \in \Lambda^\ell V^*$ is the alternating $k+\ell$-form $\omega \wedge \tau \in \Lambda^{k+\ell} V^*$ defined by*

$$(\omega \wedge \tau)(v_1, \ldots, v_{k+\ell}) := \sum_{\sigma \in S_{k,\ell}} \varepsilon(\sigma)\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)\tau\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right)$$

for $v_1, \ldots, v_{k+\ell} \in V$.

Example 7.8. The exterior product of two 1-forms $\alpha, \beta \in V^*$ is the 2-form

$$(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

The exterior product of a 1-form $\alpha \in V^*$ and a 2-form $\omega \in \Lambda^2 V^*$ is given by

$$(\alpha \wedge \omega)(u, v, w) = \alpha(u)\omega(v, w) + \alpha(v)\omega(w, u) + \alpha(w)\omega(u, v)$$

for $u, v, w \in V$.

Lemma 7.9. (i) *The exterior product is* **associative***:*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

for $\omega_1, \omega_2, \omega_3 \in \Lambda^* V^*$.

(ii) *The exterior product is* **distributive***:*

$$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

for $\omega_1, \omega_2, \omega_3 \in \Lambda^* V^*$.

(ii) *The exterior product is* **super-commutative***:*

$$\omega \wedge \tau = (-1)^{\deg(\omega)\deg(\tau)}\tau \wedge \omega$$

for $\omega, \tau \in \Lambda^* V^*$.

Proof. Let $\omega_i \in \Lambda^{k_i} V^*$, denote

$$k := k_1 + k_2 + k_3,$$

and define $S_{k_1,k_2,k_3} \subset S_k$ by

$$S_{k_1,k_2,k_3} := \left\{ \sigma \in S_k \left| \begin{array}{l} \sigma(1) < \cdots < \sigma(k_1), \\ \sigma(k_1+1) < \cdots < \sigma(k_1+k_2), \\ \sigma(k_1+k_2+1) < \cdots < \sigma(k) \end{array} \right. \right\},$$

Let $\omega \in \Lambda^k V^*$ be the alternating k-form

$$\begin{aligned} \omega(v_1, \ldots, v_k) := \sum_{\sigma \in S_{k_1,k_2,k_3}} & \varepsilon(\sigma)\omega_1 \left(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)}\right) \cdot \\ & \cdot \omega_2 \left(v_{\sigma(k_1+1)}, \ldots, v_{\sigma(k_1+k_2)}\right) \omega_3 \left(v_{\sigma(k_1+k_2+1)}, \ldots, v_{\sigma(k)}\right). \end{aligned}$$

Then it follows from Definition 7.7 that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

This proves (i). Assertion (ii) is obvious.

To prove (iii) we define the bijection

$$S_{k,\ell} \to S_{\ell,k} : \sigma \mapsto \widetilde{\sigma}$$

by

$$\widetilde{\sigma}(i) := \begin{cases} \sigma(k+i), & \text{for } i = 1, \ldots, \ell, \\ \sigma(i-\ell), & \text{for } i = \ell+1, \ldots, \ell+k. \end{cases}$$

Then

$$\varepsilon(\widetilde{\sigma}) = (-1)^{k\ell} \varepsilon(\sigma)$$

and hence, for $\omega \in \Lambda^k V^*$, $\tau \in \Lambda^\ell V^*$, and $v_1, \ldots, v_{k+\ell} \in V$, we have

$$\begin{aligned} & (\omega \wedge \tau)(v_1, \ldots, v_{k+\ell}) \\ & = \sum_{\sigma \in S_{k,\ell}} \varepsilon(\sigma)\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \tau\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right) \\ & = (-1)^{k\ell} \sum_{\widetilde{\sigma} \in S_{\ell,k}} \varepsilon(\widetilde{\sigma})\omega\left(v_{\widetilde{\sigma}(\ell+1)}, \ldots, v_{\widetilde{\sigma}(\ell+k)}\right) \tau\left(v_{\widetilde{\sigma}(1)}, \ldots, v_{\widetilde{\sigma}(\ell)}\right) \\ & = (-1)^{k\ell} (\tau \wedge \omega)(v_1, \ldots, v_{k+\ell}). \end{aligned}$$

This proves (iii) and the lemma. □

Exercise 7.10. The **Determinant Theorem** asserts that

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) & \cdots & \alpha_1(v_k) \\ \alpha_2(v_1) & \alpha_2(v_2) & \cdots & \alpha_2(v_k) \\ \vdots & \vdots & & \vdots \\ \alpha_k(v_1) & \alpha_k(v_2) & \cdots & \alpha_k(v_k) \end{pmatrix} \tag{7.2}$$

for all $\alpha_1, \ldots, \alpha_k \in V^*$ and $v_1, \ldots, v_k \in V$. Prove this formula and deduce that

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for $I = (i_1, \ldots, i_k) \in \mathcal{I}_k$, where $dx^I \in \Lambda^k(\mathbb{R}^m)^*$ is given by (7.1).

7.1.3 Pullback

Let W be an n-dimensional real vector space and $\Phi : V \to W$ be a linear map.

Definition 7.11. *The* **pullback** *of an alternating k-form $\omega \in \Lambda^k W^*$ under Φ is the alternating k-form $\Phi^*\omega \in \Lambda^k V^*$ defined by*

$$(\Phi^*\omega)(v_1, \ldots, v_k) := \omega(\Phi v_1, \ldots, \Phi v_k)$$

for $v_1, \ldots, v_k \in V$.

Lemma 7.12. (i) *The map $\Lambda^* W \to \Lambda^* V : \omega \mapsto \Phi^*\omega$ is linear and preserves the exterior product, i.e. for all $\omega \in \Lambda^k W^*$ and $\tau \in \Lambda^\ell W^*$ we have*

$$\Phi^*(\omega \wedge \tau).$$

(ii) *If $\Psi : W \to Z$ is another linear map with values in a real vector space Z then, for every $\omega \in \Lambda^k Z^*$, we have*

$$(\Psi \circ \Phi)^*\omega = \Phi^*\Psi^*\omega$$

Moreover, for every $\omega \in \Lambda^k V^*$, we have $id^*\omega = \omega$, where $\mathrm{id} : V \to V$ denotes the identity map.

(iii) *If $\Phi : V \to V$ is an endomorphism of an m-dimensional real vector space V and $\omega \in \Lambda^m V^*$ then*

$$\Phi^*\omega = \det(\Phi)\omega.$$

Proof. Assertions (i) and (ii) follow directly from the definitions. By (ii) it suffices to prove (iii) for $V = \mathbb{R}^m$. In this case assertion (iii) can be written in the form

$$\Phi^*\left(dx^1 \wedge \cdot \wedge dx^m\right) = \det(\Phi) dx^1 \wedge \cdot \wedge dx^m$$

for $\Phi \in \mathbb{R}^{m \times m}$ and this follows from (7.1) and the product formula for the determinant. This proves the lemma. □

7.2 Differential Forms

7.2.1 Definitions and Remarks

Let M be a smooth m-manifold and k be a nonnegative integer. A **differential k-form** on M is a collection of alternating k-forms

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R},$$

one for each $p \in M$, such that, for every k-tuple of smooth vector fields $X_1, \ldots, X_k \in \mathrm{Vect}(M)$, the function

$$M \to \mathbb{R} : p \mapsto \omega_p(X_1(p), \ldots, X_k(p))$$

is smooth. The set of differential k-forms on M will be denoted by $\Omega^k(M)$. A differential form $\omega \in \Omega^k(M)$ is said to have **compact support** if the set

$$\mathrm{supp}(\omega) := \overline{\{p \in M \,|\, \omega_p \neq 0\}}$$

(called the **support** of ω) is compact. The set of compactly supported k-forms on M will be denoted by $\Omega^k_c(M) \subset \Omega^k(M)$. As before we call the integer $k =: \deg(\omega)$ the **degree** of $\omega \in \Omega^k(M)$.

Remark 7.13. The set

$$\Lambda^k T^*M := \Big\{(p, \omega) \,|\, p \in M,\, \omega \in \Lambda^k T^*M\Big\}$$

is a vector bundle over M. This concept will be discussed in detail in Section 9.1. We remark here that $\Lambda^k T^*M$ admits the structure of a smooth manifold, the obvious projection $\pi : \Lambda^k T^*M \to M$ is a smooth submersion, each fiber $\Lambda^k T^*_pM$ is a vector space, and addition and scalar multiplication define smooth maps. The manifold structure is uniquely determined by the fact that each differential k-form $\omega \in \Omega^k(M)$ defines a smooth map

$$M \to \Lambda^k T^*M : p \mapsto (p, \omega_p),$$

still denoted by ω. Its composition with π is the identity on M and such a map is called a smooth section of the vector bundle. Thus $\Omega^k(M)$ can be identified the space of smooth sections of $\Lambda^k T^*M$. It is a vector space and is infinite dimensional (unless M is a finite set or $k > \dim M$). In particular, $\Lambda^0 T^*M = M \times \mathbb{R}$ and

$$\Omega^0(M) = \{f : M \to \mathbb{R} \,|\, f \text{ is smooth}\}$$

is the set of smooth real valued functions on M, also denoted by $\mathcal{F}(M)$, $C^\infty(M, \mathbb{R})$, or simply $C^\infty(M)$.

The pointwise **exterior product** defines a bilinear map

$$\Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M) : (\omega, \tau) \mapsto \omega \wedge \tau,$$

given by

$$(\omega \wedge \tau)_p := \omega_p \wedge \tau_p \tag{7.3}$$

for $p \in M$. If $f : M \to N$ is a smooth map between manifolds and $\omega \in \Omega^k(N)$ is a differential k-form on N, its **pullback** under f is the differential k-form $f^*\omega \in \Omega^k(M)$ defined by

$$(f^*\omega)_p(v_1, \dots, v_k) := \omega_{f(p)}(df(p)v_1, \dots, df(p)v_k) \tag{7.4}$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$. The next lemma summarizes the basic properties of the exterior product and pullback of differential forms.

Lemma 7.14. *Let $\phi : M \to N$ and $\psi : N \to P$ be smooth maps between manifolds.*

(i) *For all $\omega_1, \omega_2, \omega_3 \in \Omega^*(M)$ we have*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

(ii) *For all $\omega_1 \in \Omega^k(M)$ and $\omega_2, \omega_3 \in \Omega^\ell(M)$ we have*

$$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3.$$

(iii) *For $\omega, \tau \in \Omega^*(M)$ we have*

$$\omega \wedge \tau = (-1)^{\deg(\omega)\deg(\tau)} \tau \wedge \omega.$$

(iv) *The map $\Omega^k(N) \to \Omega^k(M) : \omega \mapsto \phi^*\omega$ is linear and preserves the exterior product, i.e. for $\omega \in \Omega^k(N)$ and $\tau \in \Omega^\ell(N)$ we have*

$$\phi^*(\omega \wedge \tau) = \phi^*\omega \wedge \phi^*\tau.$$

(v) *For every $\omega \in \Omega^k(P)$ we have*

$$(\psi \circ \phi)^*\omega = \phi^*\psi^*\omega.$$

Moreover, for every $\omega \in \Omega^k(M)$, we have $\mathrm{id}^*\omega = \omega$, where $\mathrm{id} : M \to M$ denotes the identity map.

(vi) *If $\phi : M \to N$ is a diffeomorphism then, for all $\omega \in \Omega^k(N)$ and $X_1, \dots, X_k \in \mathrm{Vect}(N)$, we have*

$$(\phi^*\omega)(\phi^*X_1, \dots, \phi^*X_k) = \omega(X_1, \dots, X_k) \circ \phi.$$

Proof. Assertions (i), (ii) and (iii) follow from Lemma 7.9, assertion (iv) follows from Lemma 7.12, (v) follows from Lemma 7.12 and the chain rule, and (vi) follows directly from the definitions. □

7.2.2 Differential Forms in Local Coordinates

Let M be an m-dimensional manifold equipped with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in A}$. Thus the U_α form an open cover of M and each map $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism onto an open subset of $\mathbb{R}^m$ (or of the upper half space $\mathbb{H}^m$ in case M has a nonempty boundary) such that the transition maps

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are smooth. In this situation every differential k-form $\omega \in \Omega^k(M)$ determines a family of differential k-forms $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$, one for each $\alpha \in A$, such that the restriction of ω to U_α (denoted by $\omega|_{U_\alpha}$ and defined as the pullback of ω under the inclusion of U_α into M) is given by

$$\omega|_{U_\alpha} = \phi_\alpha^*\omega_\alpha \tag{7.5}$$

for every $\alpha \in A$. Explicitly, if

$$p \in U_\alpha, \qquad v_i \in T_pM, \qquad x := \phi_\alpha(p), \qquad \xi_i := d\phi_\alpha(p)v_i$$

for $i = 1,\dots,k$ then

$$\omega_\alpha(x;\xi_1,\dots,\xi_k) = \omega_p(v_1,\dots,v_k). \tag{7.6}$$

Recall that $v_i \in T_pM$ and $\xi_i \in \mathbb{R}^m$ are related by $v_i = [\alpha,\xi_i]_p$ in the tangent space model

$$T_pM = \bigcup_{p\in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \sim .$$

Now let $e_1,\dots e_m$ denote the standard basis of $\mathbb{R}^m$ and define

$$f_{\alpha,I} : U_\alpha \to \mathbb{R}$$

by

$$f_{\alpha,I}(x) := \omega_\alpha\left(x; e_{i_1},\dots,e_{i_k}\right) = \omega_p\left([\alpha,e_{i_1}]_p,\dots,[\alpha,e_{i_k}]_p\right)$$

for $x \in \phi_\alpha(U_\alpha)$, $p := \phi_\alpha^{-1}(x) \in U_\alpha$, and $I = (i_1,\dots,i_k) \in \mathcal{I}_k$. Then $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$ can be written in the form

$$\omega_\alpha = \sum_{I\in\mathcal{I}_k} f_{\alpha,I}\,dx^I. \tag{7.7}$$

Remark 7.15. The differential forms $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$ in local coordinates satisfy the equation

$$\omega_\alpha|_{\phi_\alpha(U_\alpha\cap U_\beta)} = \left(\phi_\beta \circ \phi\alpha^{-1}\right)^* \omega_\beta|_{\phi_\beta(U_\alpha\cap U_\beta)} \tag{7.8}$$

for all $\alpha, \beta \in A$. Conversely, every family of differential k-forms $\phi_\alpha \in \Omega^k(\phi_\alpha(U_\alpha)$ that satisfy (7.8) for all $\alpha, \beta \in A$ determine a unique differential k-form $\omega \in \Omega^k(M)$ such that (7.5) holds for every $\alpha \in A$.

7.2.3 The Exterior Differential on Euclidean Space

Let $U \subset \mathbb{R}^m$ be an open set. The exterior differential is a linear operator

$$d : \Omega^k(U) \to \Omega^{k+1}(U).$$

We give two definitions of this operator, corresponding to the two ways of writing a differential form.

Definition 7.16. *Let $\omega \in \Omega^k(U)$. Then ω is a smooth map*

$$\omega : U \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R}$$

such that, for every $x \in U$, the map

$$\underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R} : (\xi_1, \ldots, \xi_k) \mapsto \omega(x; \xi_1, \ldots, \xi_k)$$

is an alternating k-form on $\mathbb{R}^m$. The **exterior differential of** ω *is the $(k+1)$-form $d\omega \in \Omega^{k+1}(U)$ defined by*

$$d\omega\,(x; \xi_1, \ldots, \xi_{k+1}) := \sum_{j=1}^{k+1} (-1)^{j-1} \left.\frac{d}{dt}\right|_{t=0} \omega\left(x + t\xi_j; \xi_1, \ldots, \widehat{\xi_j}, \ldots, \xi_{k+1}\right) \tag{7.9}$$

for $x \in U$ and $\xi_1, \ldots, \xi_{k+1} \in \mathbb{R}^m$. Here the hat indicates that the jth term is deleted.

Definition 7.17. *Let $\omega \in \Omega^k(U)$ and, for $I = (i_1, \ldots, i_k) \in \mathcal{I}_k$, define $f_I : U \to \mathbb{R}$ by*

$$f_I(x) := \omega(x; e_{i_1}, \ldots, e_{i_k}).$$

Then

$$\omega = \sum_{I \in \mathcal{I}_k} f_I dx^I$$

and the **exterior differential of** ω *is the $(k+1)$-form*

$$d\omega := \sum_{I \in \mathcal{I}_k} df_I \wedge dx^I, \qquad df_I := \sum_{\nu=1}^{m} \frac{\partial f_I}{\partial x^\nu} dx^\nu. \tag{7.10}$$

Remark 7.18. Let $f \in \Omega^0(U)$ be a smooth real valued function on U. Then $df \in \Omega^1(U)$ is the usual differential of f, which assigns to each point $x \in U$ the derivative $df(x) : \mathbb{R}^m \to \mathbb{R}$ given by

$$df(x;\xi) = df(x)\xi = \lim_{t\to 0} \frac{f(x+t\xi) - f(x)}{t} = \sum_{\nu=1}^{m} \frac{\partial f}{\partial x^\nu}\xi^\nu$$

Here the last equality asserts that the derivative of f at x is given by multiplication with the Jacobi matrix. Thus

$$df = \sum_{\nu=1}^{m} \frac{\partial f}{\partial x^\nu} dx^\nu$$

and this shows that the two definitions of $df \in \Omega^1(U)$ in (7.9) and (7.10) agree for $k = 0$.

Remark 7.19. We prove that the definitions of $d\omega$ in (7.9) and (7.10) agree for all $\omega \in \Omega^k(U)$. To see this write ω is the form

$$\omega = \sum_{I\in\mathcal{I}_k} f_I dx^I, \qquad f_I : U \to \mathbb{R}.$$

Then

$$\omega(x;\xi_1,\dots,\xi_k) = \sum_{I\in\mathcal{I}_k} f_I(x) dx^I(\xi_1,\dots,\xi_k)$$

for all $x \in U$ and $\xi_1,\dots,\xi_k \in \mathbb{R}^m$. Hence, by (7.9), we have

$$\begin{aligned}
&d\omega(x;\xi_1,\dots,\xi_{k+1}) \\
&= \sum_{I\in\mathcal{I}_k}\sum_{j=1}^{k+1}(-1)^{j-1} \left.\frac{d}{dt}\right|_{t=0} f_I(x+t\xi_j)dx^I\left(\xi_1,\dots,\widehat{\xi_j},\dots,\xi_{k+1}\right) \\
&= \sum_{I\in\mathcal{I}_k}\sum_{j=1}^{k+1}(-1)^{j-1} df_I(x;\xi_j)dx^I\left(\xi_1,\dots,\widehat{\xi_j},\dots,\xi_{k+1}\right) \\
&= \sum_{I\in\mathcal{I}_k}(df_I \wedge dx^I)(x;\xi_1,\dots,\xi_{k+1})
\end{aligned}$$

for all $x \in U$ and $\xi_1,\dots,\xi_{k+1} \in \mathbb{R}^m$. The last term agrees with the right hand side of (7.10).

Lemma 7.20. *Let $U \subset \mathbb{R}^m$ be an open set.*
(i) *The exterior differential $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is a linear operator.*
(ii) *The exterior differential satisfies the* **Leibnitz rule**

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge d\tau.$$

(iii) *The exterior differential satisfies $d \circ d = 0$.*
(iv) *The exterior differential commutes with pullback: If $\phi : U \to V$ is a smooth map to an open subset $V \subset \mathbb{R}^n$ then, for every $\omega \in \Omega^k(V)$, we have*

$$\phi^* d\omega = d\phi^* \omega.$$

Proof. Assertion (i) is obvious. To prove (ii) it suffices to consider two differential forms

$$\omega = f dx^I, \qquad \tau = g dx^J$$

with $I = (i_1, \ldots, i_k) \in \mathcal{I}_k$, $J = (j_1, \ldots, j_\ell) \in \mathcal{I}_\ell$, and $f, g : U \to \mathbb{R}$. Then it follows from Definition 7.17 that

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) \\ &= d(fg) \wedge dx^I \wedge dx^J \\ &= \bigl(gdf + fdg\bigr) \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (gdx^J) + (-1)^k (f dx^I) \wedge (dg \wedge dx^J) \\ &= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \end{aligned}$$

For general differential forms on U assertion assertion (ii) follows from the special case and (i).

We prove (iii). For $f \in \Omega^0(U)$ we have

$$ddf = d\left(\sum_{j=1}^m \frac{\partial f}{\partial x_j} dx^j\right) = \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0.$$

Here the last equality follows from the fact that the second partial derivatives commute. This implies that, for every smooth function $f : U \to \mathbb{R}$ and every multi-index $I = (i_1, \ldots, i_k) \in \mathcal{I}_k$, we have

$$dd(f dx^I) = d(df \wedge dx^I) = ddf \wedge dx^I - df \wedge ddx^I = 0.$$

Here the second equation follows from (ii) and the last from the fact that $ddf = 0$ (as showsn above) and $ddx^I = 0$ (by definition). With this understood assertion (iii) follows from the fact that $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is a linear operator.

We prove (iv). Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets and $\phi : U \to V$ be a smooth map. We denote the elements of U by $x = (x^1, \dots, x^m)$, the elements of V by $y = (y^1, \dots, y^n)$, and the coordinates of $\phi(x)$ by

$$\phi(x) =: (\phi^1(x), \dots, \phi^n(x))$$

for $x \in U$. Thus each ϕ^j s a smooth map from U to $\mathbb{R}$ and we have

$$\phi^* dy^j = \sum_{i=1}^m \frac{d\phi^j}{dx^i} dx^i = d\phi^j. \tag{7.11}$$

Moreover, if $g \in \Omega^0(V)$ is a smooth real valued function on V, then

$$\phi^* g = g \circ \phi, \qquad dg = \sum_{j=1}^n \frac{\partial g}{\partial y^j} dy^j,$$

and hence

$$\begin{aligned} d(\phi^* g) &= \sum_{i=1}^m \frac{\partial (g \circ \phi)}{\partial x^i} dx^i \\ &= \sum_{i=1}^m \sum_{j=1}^n \left(\frac{\partial g}{\partial y^j} \circ \phi \right) \frac{\partial \phi^j}{\partial x^i} dx^i \\ &= \sum_{j=1}^n \left(\frac{\partial g}{\partial y^j} \circ \phi \right) d\phi^j \\ &= \sum_{j=1}^n \left(\frac{\partial g}{\partial y^j} \circ \phi \right) \phi^* dy^j \\ &= \phi^* dg. \end{aligned} \tag{7.12}$$

Here the second equation follows from the chain rule and the fourth equation follows from (7.11). For $J = (j_1, \dots, j_k) \in \mathcal{I}_k$ we have

$$d(\phi^* dy^J) = d(\phi^* dy^{j_1} \wedge \cdots \phi^* dy^{j_k}) = d(d\phi^{j_1} \wedge \cdots \wedge d\phi^{j_k}) = 0. \tag{7.13}$$

Here the first equation follows from Lemma 7.14 and the determinant theorem in Exercise 7.10, the second equation follows from (7.11), and the last equation follows from the Leibnitz rule in (ii) and the fact that $dd\phi^j = 0$ for every j, by (iii). Combining (7.12) and (7.13) we obtain

$$\phi^* d(g dy^J) = \phi^* dg \wedge \phi^* dy^J = d(\phi^* g) \wedge \phi^* dy^J = d\phi^*(g dy^J)$$

for every smooth function $g : V \to \mathbb{R}$ and every $J \in \mathcal{I}_k$. This proves (iv) and the lemma. □

7.2.4 The Exterior Differential on Manifolds

Let M be a smooth m-dimensional manifold with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ and let $\omega \in \Omega^k(M)$ be a differential k-form on M. Denote by

$$\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$$

the corresponding differential forms in local coordinates so that

$$\omega|_{U_\alpha} = \phi_\alpha^* \omega_\alpha \tag{7.14}$$

for every $\alpha \in A$. The **exterior differential** of ω is defined as the unique $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$ that satisfies

$$d\omega|_{U_\alpha} = \phi_\alpha^* d\omega_\alpha \tag{7.15}$$

for every $\alpha \in A$. To see that such a form exists we observe that the ω_α satisfy equation (7.8) for all $\alpha, \beta \in A$. Then, by Lemma 7.20, we have

$$d\omega_\alpha|_{\phi_\alpha(U_\alpha \cap U_\beta)} = (\phi_\beta \circ \phi_\alpha^{-1})^* d\omega_\beta|_{\phi_\beta(U_\alpha \cap U_\beta)}$$

for all $\alpha, \beta \in A$ and so the existence and uniqueness of the $(k+1)$-form $d\omega$ satisfying (7.15) follows from Remark 7.15. It also follows from Lemma 7.20 that this definition of $d\omega$ is independent of the choice of the atlas.

Lemma 7.21. *Let M be a smooth manifold.*

(i) *The exterior differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear operator.*

(ii) *The exterior differential satisfies the* **Leibnitz rule**

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge d\tau.$$

(iii) *The exterior differential satisfies $d \circ d = 0$.*

(iv) *The exterior differential commutes with pullback: If $\phi : M \to N$ is a smooth map between manifolds then, for every $\omega \in \Omega^k(N)$, we have*

$$\phi^* d\omega = d\phi^* \omega.$$

Proof. This follows immediately from Lemma 7.20 and the definitions. □

7.2.5 DeRham Cohomology

Lemma 7.21 shows that there is a cochain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M),$$

called the **deRham complex**. A differential form $\omega \in \Omega^k(M)$ is called **closed** if $d\omega = 0$ and is called **exact** if there is a $(k-1)$-form $\tau \in \Omega^{k-1}(M)$ such that $d\tau = \omega$. Lemma 7.21 (iii) asserts that every exact k-form is closed and the quotient space

$$H^k(M) := \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\mathrm{im}\ d : \Omega^{k-1}(M) \to \Omega^k(M)} = \frac{\{\text{closed } k - \text{forms on } M\}}{\{\text{exact } k - \text{forms on } M\}}$$

is called the kth **deRham cohomology group** of M. By Lemma 7.21 (i) is a real vector space. By Lemma 7.21 (ii) the exterior product defines a bilinear map

$$H^k(M) \times H^\ell(M) \to H^{k+\ell}(M) : ([\omega], [\tau]) \mapsto [\omega] \cup [\tau] := [\omega \wedge \tau]$$

called the **cup product**. By Lemma 7.14 (iv) the pullback by a smooth map $\phi : M \to N$ induces a homomorphism

$$\phi^* : H^k(N) \to H^k(M).$$

By Lemma 7.14 this map is linear and preserves the cup product.

Example 7.22. The deRham cohomology group $H^0(M)$ is the space of smooth functions $f : M \to \mathbb{R}$ whose differential vanishes everywhere. Thus $H^0(M)$ is the space of locally constant real valued functions on M. If M is connected the evaluation map at any point defines an isomorphism

$$H^0(M) = \mathbb{R}.$$

To gain a better understanding of the deRham cohomology groups we introduce the integral of a differential form of maximal degree over a compact oriented manifold, prove the theorem of Stokes, and examine the formula of Cartan for the Lie derivative of a differential form in the direction of a vector field. These topics will be discussed in the next two sections.

7.3 Integration

7.3.1 Definition of the Integral

Let M be an oriented m-manifold, with or without boundary and not necessarily compact. Let $\{U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be an oriented atlas on M. Thus the U_α form an open cover of M and the

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$$

are homeomorphisms onto open subsets $\phi_\alpha(U_\alpha) \subset \mathbb{H}^m$ of the upper half space

$$\mathbb{H}^m := \{x \in \mathbb{R}^m \,|\, x^m \geq 0\}$$

such that the transition maps

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are smooth and

$$\det\left(d\phi_{\beta\alpha}(x)\right) > 0$$

for all $\alpha, \beta \in A$ and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. Choose a partition of unity

$$\rho_\alpha : M \to [0,1], \qquad \alpha \in A,$$

subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. Thus each point $p \in M$ has a neighborhood on which only finitely many of the ρ_α do not vanish and

$$\mathrm{supp}(\rho_\alpha) \subset U_\alpha, \qquad \sum_\alpha \rho_\alpha \equiv 1.$$

Definition 7.23. *Let $\omega \in \Omega_c^m(M)$ be a differential form with compact support and, for $\alpha \in A$, let*

$$\omega_\alpha \in \Omega^m(\phi_\alpha(U_\alpha)), \qquad g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}$$

be given by

$$\omega|_{U_\alpha} =: \phi_\alpha^* \omega_\alpha, \qquad \omega_\alpha =: g_\alpha(x) dx^1 \wedge \cdots \wedge dx^m.$$

The **integral of** ω **over** M *is the real number*

$$\int_M \omega := \sum_{\alpha\in A} \int_{\phi_\alpha(U_\alpha)} \rho_\alpha(\phi_\alpha^{-1}(x)) g_\alpha(x) dx^1 \cdots dx^m \tag{7.16}$$

The sum on the right is finite because only finitely many of the products $\rho_\alpha\omega$ are nonzero. (Prove this!)

Lemma 7.24. *The integral of ω over M is independent of the oriented atlas and the partition of unity used to define it.*

Proof. Choose another atlas $\{V_\beta, \psi_\beta\}_{\beta\in B}$ on M and a partition of unity $\theta_\beta : M \to [0,1]$ subordinate to the cover $\{V_\beta\}_{\beta\in B}$. For $\beta \in B$ define

$$\omega_\beta \in \Omega^m(\psi_\beta(V_\beta)), \qquad h_\beta : \psi_\beta(V_\beta) \to \mathbb{R}$$

by

$$\omega|_{V_\beta} =: \psi_\beta^*\omega_\beta, \qquad \omega_\beta =: h_\beta(y)dy^1 \wedge \cdots \wedge dy^m.$$

Then it follows from Lemma 7.12 (iv) that

$$g_\alpha(x) = h_\beta\left(\psi_\beta \circ \phi_\alpha^{-1}(x)\right) \underbrace{\det\left(d(\psi_\beta \circ \phi_\alpha^{-1})(x)\right)}_{>0} \tag{7.17}$$

for every $x \in \phi_\alpha(U_\alpha \cap V_\beta)$. Hence

$$\begin{aligned}
\int_M \omega &= \sum_{\alpha\in A} \int_{\phi_\alpha(U_\alpha)} (\rho_\alpha \circ \phi_\alpha^{-1}) g_\alpha dx^1 \cdots dx^m \\
&= \sum_\alpha \sum_\beta \int_{\phi_\alpha(U_\alpha\cap V_\beta)} (\rho_\alpha \circ \phi_\alpha^{-1})(\theta_\beta \circ \phi_\alpha^{-1}) g_\alpha dx^1 \cdots dx^m \\
&= \sum_\alpha \sum_\beta \int_{\psi_\beta(U_\alpha\cap V_\beta)} (\rho_\alpha \circ \psi_\beta^{-1})(\theta_\beta \circ \psi_\beta^{-1}) h_\beta dy^1 \cdots dy^m \\
&= \sum_\beta \int_{\psi_\beta(V_\beta)} (\theta_\beta \circ \psi_\beta^{-1}) h_\beta dy^1 \cdots dy^m.
\end{aligned}$$

Here the first equation is the definition of the integral, the second equation follows from the fact that the θ_β form a partition of unity, the third equation follows from (7.17) and the change of variables formula, and the last equation follows from the fact that the ρ_α form a partition of unity. This proves the lemma. □

We can think of the integral is a functional

$$\Omega_c^m(M) \to \mathbb{R} : \omega \mapsto \int_M \omega.$$

It follows directly from the definition that this functional is linear.

Exercise 7.25. If $f : M \to N$ is an orientation preserving diffeomorphism between oriented m-manifolds then $\int_M f^*\omega = \int_N \omega$ for every $\omega \in \Omega_c^m(N)$. If $f : M \to N$ is an orientation reversing diffeomorphism between oriented m-manifolds then $\int_M f^*\omega = -\int_N \omega$ for every $\omega \in \Omega_c^m(N)$.

7.3.2 The Theorem of Stokes

Theorem 7.26. *Let M be an oriented m-manifold with boundary and let $\omega \in \Omega_c^{m-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. The proof has three steps.

Step 1. *The theorem holds for $M = \mathbb{H}^m$.*

The boundary of $\mathbb{H}^m = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m \,|\, x^m \geq 0\}$ is the subset $\partial\mathbb{H}^m = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m \,|\, x^m = 0\}$, diffeomorphic to $\mathbb{R}^{m-1}$. Consider the differential $(m-1)$-form

$$\omega = \sum_{i=1}^m g_i(x) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m$$

where the $g_i : \mathbb{H}^m \to \mathbb{R}$ are smooth functions with compact support (in the closed upper half space) and the hat indicates that the ith term is deleted in the ith summand. Then

$$\begin{aligned} d\omega &= \sum_{i=1}^m \frac{\partial g_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m \\ &= \sum_{i=1}^m (-1)^{i-1} \frac{\partial g_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

Choose $R > 0$ so large that the support of each g_i is contained in the set $[-R, R]^{m-1} \times [0, R]$. Then

$$\begin{aligned} \int_{\mathbb{H}^m} d\omega &= \sum_{i=1}^m (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial g_i}{\partial x^i}(x^1, \ldots, x^m) dx^1 \cdots dx^m \\ &= (-1)^{m-1} \int_{-R}^R \cdots \int_{-R}^R \int_0^R \frac{\partial g_m}{\partial x^m}(x^1, \ldots, x^m) dx^m dx^1 \cdots dx^{m-1} \\ &= (-1)^m \int_{-R}^R \cdots \int_{-R}^R g_m(x^1, \ldots, x^{m-1}, 0) dx^1 \cdots dx^{m-1} \\ &= \int_{\partial\mathbb{H}^m} \omega \end{aligned}$$

Here the second equation follows from Fubini's theorem, the third equation follows again from the fundamental theorem of calculus. To understand the last equation we observe that the restriction of ω to the boundary is

$$\omega|_{\partial\mathbb{H}^m} = g_m(x^1, \ldots, x^{m-1}, 0)\, dx^1 \wedge \cdots \wedge dx^{m-1}.$$

Moreover, the orientation of $\mathbb{R}^{m-1}$ as the boundary of $\mathbb{H}^m$ is $(-1)^m$ times the standard orientation of $\mathbb{R}^{m-1}$ because the outward pointing unit normal vector at any boundary point is $\nu = (0, \dots, 0, -1)$. This proves the last equation above and completes the proof of Step 1.

Step 2. *We prove the theorem for every differential* $(m-1)$*-form whose support is compact and contained in a coordinate chart.*

Let $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{H}^m$ be a coordinate chart and $\omega \in \Omega_c^{m-1}(M)$ be a compactly supported differential form with

$$\mathrm{supp}(\omega) \subset U_\alpha.$$

Define $\omega_\alpha \in \Omega^{m-1}(\phi_\alpha(U_\alpha))$ by

$$\omega|_{U_\alpha} =: \phi_\alpha^* \omega_\alpha$$

and extend ω_α to all of $\mathbb{H}^m$ by setting ω_α equal to zero on $\mathbb{H}^m \setminus \phi_\alpha(U_\alpha)$. Since $\phi_\alpha(U_\alpha \cap \partial M) = \phi_\alpha(U_\alpha) \cap \partial \mathbb{H}^m$ we obtain, using Step 1, that

$$\begin{aligned}
\int_M d\omega &= \int_{U_\alpha} d\phi_\alpha^* \omega_\alpha \\
&= \int_{U_\alpha} \phi_\alpha^* d\omega_\alpha \\
&= \int_{\phi_\alpha(U_\alpha)} d\omega_\alpha \\
&= \int_{\phi_\alpha(U_\alpha) \cap \partial \mathbb{H}^m} \omega_\alpha \\
&= \int_{U_\alpha \cap \partial M} \phi_\alpha^* \omega_\alpha \\
&= \int_{\partial M} \omega.
\end{aligned}$$

This proves Step 2.

Step 3. *We prove the theorem.*

Choose an atlas $\{U_\alpha, \phi_\alpha\}_\alpha$ and a partition of unity $\rho_\alpha : M \to [0, 1]$ subordinate to the cover $\{U_\alpha\}_\alpha$. Then, by Step 2, we have

$$\int_M d\omega = \sum_\alpha \int_M d(\rho_\alpha \omega) = \sum_\alpha \int_{\partial M} \rho_\alpha \omega = \int_{\partial M} \omega.$$

This proves Step 3 and the theorem. □

7.3.3 Examples

Example 7.27. Let $U \subset \mathbb{R}^2$ be a bounded open set with connected smooth boundary $\Gamma := \partial U$ and choose an embedded loop $\mathbb{R}/\mathbb{Z} \to \Gamma : t \mapsto (x(t), y(t))$ parametrizing Γ. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions and consider the 1-form

$$\omega = f dx + g dy \in \Omega^2(\mathbb{R}^2).$$

Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

and hence, by Stokes' theorem, we have

$$\begin{aligned} \int_U \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy &= \int_\Gamma (f dx + g dy) \\ &= \int_0^1 \left(f(x(t), y(t))\dot{x}(t) + g(x(t), y(t))\dot{y}(t) \right) dt. \end{aligned}$$

Example 7.28. Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional embedded surface and $\nu : \Sigma \to S^2$ be a Gauss map. Thus $\nu(x) \perp T_x\Sigma$ for every $x \in \Sigma$. Define the 2-form $\mathrm{dvol}_\sigma \in \Omega^2(\Sigma)$ by

$$\mathrm{dvol}_\Sigma(x; v, w) := \det(\nu(x), v, w)$$

for $x \in \Sigma$ and $v, w \in T_x\Sigma$. In other words

$$\mathrm{dvol}_\Sigma = \nu^1 dx^2 \wedge dx^3 + \nu^2 dx^3 \wedge dx^1 + \nu^3 dx^1 \wedge dx^2$$

and

$$\nu^1 \mathrm{dvol}_\Sigma = dx^2 \wedge dx^3, \quad \nu^2 \mathrm{dvol}_\Sigma = dx^3 \wedge dx^1, \quad \nu^3 \mathrm{dvol}_\Sigma = dx^1 \wedge dx^2.$$

Let $u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth map and consider the 1-form

$$\omega = u_1 dx^1 + u_2 dx^2 + u_3 dx^3 \in \Omega^1(\Sigma).$$

Its exterior differential is

$$d\omega = \langle \mathrm{curl}(u), \nu \rangle \, \mathrm{dvol}_\Sigma, \qquad \mathrm{curl}(u) := \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix},$$

and hence Stokes' theorem gives the identity

$$\int_\Sigma \langle \mathrm{curl}(u), \nu \rangle \mathrm{dvol}_\Sigma = \int_{\partial\Sigma} \sum_{i=1}^3 u_i dx^i.$$

Example 7.29. If M is an oriented m-manifold without boundary and $\tau \in \Omega_c^{m-1}(M)$ is a compactly supported $(m-1)$-form it follows from Stokes' theorem that

$$\int_M d\tau = 0.$$

We prove in the next section that, when M is connected, the converse holds as well: if $\omega \in \Omega_c^m(M)$ is compactly supported m-form such that $\int_M \omega = 0$ then there is a $\tau \in \Omega_c^{m-1}(M)$ such that $d\tau = \omega$.

7.4 Cartan's Formula

Let M and N be smooth manifold, $I \subset \mathbb{R}$ be an interval, and

$$I \times M \to N : (t, p) \mapsto \phi_t(p)$$

be a smooth map. For $t \in I$ define the operator

$$h_t : \Omega^k(N) \to \Omega^{k-1}(M)$$

by

$$(h_t\omega)_p(v_1, \dots, v_{k-1}) := \omega_{\phi_t(p)}\left(\partial_t\phi_t(p), d\phi_t(p)v_1, \dots, df\phi_t(p)v_{k-1}\right) \quad (7.18)$$

for $p \in M$ and $v_1, \dots, v_{k-1} \in T_pM$.

Theorem 7.30 (Cartan). *For every $\omega \in \Omega^k(N)$ we have*

$$\frac{d}{dt}\phi_t^*\omega = dh_t\omega + h_t d\omega. \quad (7.19)$$

Proof. The proof has four steps.

Step 1. *Equation* (7.19) *holds for $k = 0$.*

Let $f : N \to \mathbb{R}$ be a smooth function. Then

$$\frac{d}{dt}(\phi_t^* f)(p) = \frac{d}{dt}f(\phi_t(p)) = df(\phi_t(p))\partial_t\phi_t(p) = h_t df(p)$$

as claimed.

Step 2. *Equation* (7.19) *holds for $k = 1$.*

Assume first that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. Let

$$I \times \mathbb{R}^m \to \mathbb{R}^n : (t, x) \mapsto \phi_t(x) = (\phi_t^1(x), \dots, \phi_t^n(x))$$

be a smooth map and

$$\beta = \sum_{\nu=1}^{n} g_\nu dy^\nu$$

be a smooth 1-form on $\mathbb{R}^n$, where $g_\nu : \mathbb{R}^n \to \mathbb{R}$ is a smooth function for $\nu = 1, \dots, n$. Then

$$d\beta = \sum_{\mu,\nu=1}^{n} \frac{\partial g_\nu}{\partial y^\mu} dy^\mu \wedge dy^\nu, \qquad h_t\beta = \sum_{\nu=1}^{n} (g_\nu \circ \phi_t)\partial_t \phi_t^\nu,$$

$$dh_t\beta = \sum_{i=1}^{m} \sum_{\mu,\nu=1}^{n} \left(\frac{\partial g_\nu}{\partial y^\mu} \circ \phi_t\right) \frac{\partial \phi_t^\mu}{\partial x^i} \partial_t \phi_t^\nu dx^i + \sum_{i=1}^{m}\sum_{\nu=1}^{n} (g_\nu \circ \phi_t) \frac{\partial^2 \phi_t^\nu}{\partial t \partial x^i} dx^i,$$

$$h_t d\beta = \sum_{i=1}^{m} \sum_{\mu,\nu=1}^{n} \left(\frac{\partial g_\nu}{\partial y^\mu} \circ \phi_t\right) \left(\partial_t \phi_t^\mu \frac{\partial \phi_t^\nu}{\partial x^i} - \partial_t \phi_t^\nu \frac{\partial \phi_t^\mu}{\partial x^i}\right) dx^i.$$

Moreover,

$$\phi_t^*\beta = \sum_{i=1}^{m}\sum_{\nu=1}^{n} (g_\nu \circ \phi_t) \frac{\partial \phi_t^\nu}{\partial x^i} dx^i$$

and hence

$$\begin{aligned} \frac{d}{dt}\phi_t^*\beta &= \sum_{i=1}^{m}\sum_{\nu=1}^{n} \frac{d}{dt}\left((g_\nu \circ \phi_t)\frac{\partial \phi_t^\nu}{\partial x^i}\right) dx^i \\ &= \sum_{i=1}^{m} \sum_{\mu,\nu=1}^{n} \left(\frac{\partial g_\nu}{\partial y^\mu} \circ \phi_t\right) \partial_t \phi_t^\mu \frac{\partial \phi_t^\nu}{\partial x^i} dx^i + \sum_{i=1}^{m}\sum_{\nu=1}^{n} (g_\nu \circ \phi_t) \frac{\partial^2 \phi_t^\nu}{\partial t \partial x^i} dx^i \\ &= dh_t\beta + h_t d\beta \end{aligned}$$

as claimed. This proves Step 2 for $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. The general case can be reduced to this by choosing local coordinates.

Step 3. *The operator $h_t : \Omega^*(M) \to \Omega^{*-1}(M)$ is linear and satisfies*

$$h_t(\omega \wedge \tau) = h_t\omega \wedge \phi_t^*\tau + (-1)^{\deg(\omega)} \phi_t^*\omega \wedge h_t\tau$$

for all $\omega, \tau \in \Omega^*(M)$.

This follows directly from the definitions.

Step 4. *Equation (7.19) holds for every $\omega \in \Omega^k(M)$ and every $k \geq 0$.*

We prove this by induction on k. For $k = 0, 1$ the assertion was proved in Steps 1 and 2. Thus assume $k \geq 2$ and suppose the assertion has been established for $k-1$. Since every k-form $\omega \in \Omega^k(N)$ can be written as a finite sum of exterior products of a 1-form and a $(k-1)$-form it suffices to assume that

$$\omega = \beta \wedge \tau, \qquad \beta \in \Omega^1(N), \qquad \tau \in \Omega^{k-1}(N).$$

In this case we have

$$\begin{aligned}
\frac{d}{dt}\phi_t^*\omega &= \frac{d}{dt}(\phi_t^*\alpha \wedge \phi_t^*\tau) \\
&= (h_t d\beta + dh_t\beta) \wedge \phi_t^*\tau + \phi_t^*\beta \wedge (dh_t\tau + h_t d\tau) \\
&= h_t(d\beta \wedge \tau) - \phi_t^* d\beta \wedge h_t\tau + d(h_t\beta \wedge \phi_t^*\tau) - h_t\beta \wedge d\phi_t^*\tau \\
&\quad - h_t(\beta \wedge d\tau) + h_t\beta \wedge \phi_t^* d\tau - d(\phi_t^*\beta \wedge h_t\tau) + d\phi_t^*\beta \wedge h_t\tau \\
&= d(h_t\beta \wedge \phi_t^*\tau - \phi_t^*\beta \wedge h_t\tau) + h_t(d\beta \wedge \tau - \beta \wedge d\tau) \\
&= dh_t\omega - h_t d\omega.
\end{aligned}$$

Here the second equation follows from Step 2 and the induction hypothesis, the third equation follows from Step 3, the fourth equation follows from the fact that the exterior derivative commutes with pullback, and the last equation follows from Step 3 and the Leibniz rule for the exterior derivative. This proves Step 4 and the theorem. □

Corollary 7.31. *Let M^m and N^n be oriented manifolds without boundary and $\phi_t : M \to M$, $0 \le t \le 1$, be a proper smooth homotopy, so that*

$$K \subset N \text{ is compact} \qquad \Longrightarrow \qquad \bigcup_t \phi_t^{-1}(K) \subset M \text{ is compact.}$$

Let $\omega \in \Omega_c^k(N)$ be closed k-form with compact support. The there is a $(k-1)$-form $\tau \in \Omega_c^{k-1}(M)$ with compact support such that

$$d\tau = \phi_1^*\omega - \phi_0^*\omega.$$

Proof. By Theorem 7.30, we have

$$\phi_1^*\omega - \phi_0^*\omega = \int_0^1 \frac{d}{dt}\phi_t^*\omega\, dt = \int_0^1 dh_t\omega\, dt = d\tau, \qquad \tau := \int_0^1 h_t\omega\, dt,$$

where $h_t : \Omega^k(M) \to \Omega^{k-1}(M)$ is given by (7.18). We have

$$\operatorname{supp}(\tau) \subset \bigcup_{0 \le t \le 1} \phi_t^{-1}(\operatorname{supp}(\omega)).$$

The set on the right is compact because the homotopy is proper. This proves the corollary. □

Let $X \in \mathrm{Vect}(M)$. The **interior product** of X with a differential form $\omega \in \Omega^k(M)$ is the $(k-1)$-form $\iota(X)\omega \in \Omega^{k-1}(M)$ defined by

$$(\iota(X)\omega)_p(v_1, \dots, v_{k-1}) := \omega_p(X(p), v_1, \dots, v_{k-1})$$

for $p \in M$ and $v_1, \dots, v_{k-1} \in T_pM$. If X is complete and $\phi_t \in \mathrm{Diff}(M)$ denotes the flow of X the **Lie derivative of** ω **in the direction of** X is defined by

$$\mathcal{L}_X\omega := \left.\frac{d}{dt}\right|_{t=0} \phi_t^*\omega.$$

This formula continues to be meaningful pointwise even if X is not complete.

Corollary 7.32 (Cartan). *The Lie derivative of* $\omega \in \Omega^k(M)$ *in the direction* $X \in \mathrm{Vect}(M)$ *is given by*

$$\mathcal{L}_X\omega = d\iota(X)\omega + \iota(X)d\omega. \tag{7.20}$$

Proof. Assume for simplicity that X is complete and let

$$\mathbb{R} \times M \to M : (t, p) \mapsto \phi_t(p)$$

denote the flow of X. Then the operator $h_t : \Omega^k(M) \to \Omega^{k-1}(M)$ in (7.18) is given by

$$h_t\omega = \phi_t^*\iota(X)\omega.$$

In particular $h_0\omega = \iota(X)\omega$ and hence (7.20) follows from (7.19) with $t = 0$. This proves the corollary. □

Corollary 7.33. *Let* $\omega \in \Omega^k(M)$ *and* $X_1, \dots, X_{k+1} \in \mathrm{Vect}(M)$. *Then*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} \mathcal{L}_{X_i}\left(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\right) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \omega\left([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\right) \end{aligned} \tag{7.21}$$

Proof. The proof is by induction on k. For $k = 0$ the equation is obvious. For $k = 1$ we use functoriality of the Lie derivative. Thus, if if ϕ_t is the flow of X, we have

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} \phi_t^* Y = -[X, Y].$$

and

$$\begin{aligned}
\mathcal{L}_X\left(\omega(Y)\right) &= \left.\frac{d}{dt}\right|_{t=0} \phi_t^*\left(\omega(Y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\phi_t^*\omega\right)\left(\phi_t^*Y\right) \\
&= (\mathcal{L}_X\omega)(Y) - \omega([X,Y]) \\
&= d\omega(X,Y) - \mathcal{L}_Y(\omega(X)) - \omega([X,Y]).
\end{aligned}$$

Here the last equality follows from Corollary 7.32. This gives

$$d\omega(X,Y) = \mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) + \omega([X,Y]) \tag{7.22}$$

as claimed. For $\omega = \alpha \wedge \tau$ with $\alpha \in \Omega^1(M)$ and $\tau \in \Omega^{k-1}(M)$ the assertion follows by induction. The induction step uses the Leibniz rule for the exterior derivative and is left to the reader. □

Example 7.34. If $\omega \in \Omega^2(M)$ and $X, Y, Z \in \mathrm{Vect}(M)$ then

$$\begin{aligned}
d\omega(X,Y,Z) = \mathcal{L}_X\big(\omega(Y,Z)\big) + \mathcal{L}_Y\big(\omega(Z,X)\big) + \mathcal{L}_Z\big(\omega(X,Y)\big) \\
- \omega(X,[Y,Z]) - \omega(Y,[Z,X]) - \omega(Z,[X,Y]).
\end{aligned} \tag{7.23}$$

Exercise 7.35. Prove the formula (7.21) directly in local coordinates using Definition 7.16.

Exercise 7.36. Deduce the formula (7.20) in Corollary 7.32 from (7.21) by an induction argument, starting with $k = 1$.

Exercise 7.37. Deduce the formula (7.19) in Theorem 7.30 from (7.20). **Hint:** Assume first that $\phi_t : M \to N$ is an embedding. Then there is a smooth family of vector field $Y_t \in \mathrm{Vect}(N)$ such that

$$Y_t \circ \phi_t = \partial_t \phi_t.$$

For example, choose a Riemannian metric on N and define

$$Y_t(\exp_{\phi_t(p)}(v)) := \rho(|v|)d\exp_{\phi_t(p)}(v)\partial_t\phi_t(v)$$

for a suitable cutoff function ρ. Let ψ_t be isotopy of N generated by Y_t via $\partial_t\psi_t = Y_t \circ \psi_t$ and $\psi_0 = \mathrm{id}$. Then

$$\phi_t = \psi_t \circ \phi_0.$$

Now deduce (7.19) from (7.20) for $\mathcal{L}_{Y_t}\omega$. To prove (7.19) in general replace $\phi_t : M \to N$ by the embedding

$$\widetilde{\phi}_t : M \to \widetilde{N} := M \times N, \qquad \widetilde{\phi}_t(p) := (p, \phi_t(p)).$$

7.5 The Degree Theorem

7.5.1 Integration and Exactness

Theorem 7.38. *Let M be a connected oriented m-dimensional manifold without boundary and $\omega \in \Omega_c^m(M)$ be an m-form with compact support. Then the the following are equivalent.*

(i) *The integral of ω over M vanishes.*

(ii) *There is an $(m-1)$-form τ on M with compact support such that $d\tau = \omega$.*

Proof. That (ii) implies (i) follows from Stokes' Theorem 7.26. We prove in two steps that (i) implies (ii).

Step 1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function whose support is contained in the set $(a,b)^m$ where $a < b$. Then there are smooth functions $u_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$, supported in $(a,b)^m$, such that*

$$f = \sum_{i=1}^m \frac{\partial u_i}{\partial x^i}.$$

Thus

$$f dx^1 \wedge \cdots \wedge dx^m = d\left(\sum_{i=1}^m (-1)^{i-1} u_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m\right).$$

To see this, choose a smooth function $\rho : \mathbb{R} \to [0,1]$ such that

$$\rho(t) = \begin{cases} 0, & \text{for } t \leq a + \varepsilon, \\ 1, & \text{for } t \geq b - \varepsilon, \end{cases}$$

for some $\varepsilon > 0$ and define $f_i : \mathbb{R}^m \to \mathbb{R}$ by $f_0 := 0$, $f_m := f$, and

$$f_i(x) := \int_a^b \cdots \int_a^b f(x^1, \dots, x^i, \xi^{i+1}, \dots, \xi^m) \dot{\rho}(x^{i+1}) \cdots \dot{\rho}(x^m) d\xi^{i+1} \cdots d\xi^m$$

for $i = 1, \dots, m-1$. Then each f_i is supported in $(a,b)^m$. For $i = 1, \dots, m$ define $u_i : \mathbb{R}^m \to \mathbb{R}$ by

$$u_i(x) := \int_a^{x_i} (f_i - f_{i-1})(x^1, \dots, x^{i-1}, \xi, x^i, \dots, x^m)\, d\xi.$$

Then u_i is supported in $(a,b)^m$ and

$$\frac{\partial u_i}{\partial x^i} = f_i - f_{i-1}.$$

This proves Step 1.

Step 2. *We prove that (i) implies (ii).*

Choose a point $p_0 \in M$, an open neighborhood $U_0 \subset M$ of p_0, and an orientation preserving coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ such that the image of ϕ_0 is the open unit cube $(0,1)^m \subset \mathbb{R}^m$. Since M is connected and has no boundary there is, for every $p \in M$, a diffeomorphism $\psi_p : M \to M$, isotopic to the identity, such that

$$\psi_p(p_0) = p.$$

Thus the open sets

$$U_p := \psi_p(U_0)$$

cover M. Choose a partition of unity $\rho_p : M \to [0,1]$ subordinate to this cover. Since the set $K := \mathrm{supp}(\omega)$ is compact there are only finitely many points $p \in M$ such that the function ρ_p does not vanish on K. Number these points as $p_1, \dots, p_n$ and abbreviate

$$U_i := U_{p_i}, \qquad \rho_i := \rho_{p_i}, \qquad \psi_i := \psi_{p_i}$$

for $i = 1, \dots, n$. Then

$$\mathrm{supp}(\rho_i) \subset U_i = \psi_i(U_0), \qquad \sum_{i=1}^n \rho_i|_K \equiv 1.$$

Hence $\mathrm{supp}(\rho_i\omega) \subset U_i$ and

$$\mathrm{supp}(\psi_i^*(\rho_i\omega)) \subset U_0.$$

Since ψ_i is smoothly isotopic to the identity and $\phi_i\omega$ has compact support, it follows from Corollary 7.31 that there is a compactly supported $(m-1)$-form $\tau_i \in \Omega_c^{m-1}(M)$ such that

$$d\tau_i = \psi_i^*(\rho_i\omega) - \rho_i\omega.$$

Hence it follows from Stokes' theorem 7.26 that

$$\int_M \sum_{i=1}^n \psi_i^*(\rho_i\omega) = \int_M \sum_{i=1}^n \rho_i\omega = \int_M \omega = 0.$$

Now $\psi_i^*(\rho_i\omega)$ is supported in $\psi_i^{-1}(U_i) = U_0$ and so is $\sum_{i=1}^n \psi_i^*(\rho_i\omega)$. Thus the pushforward of this sum under the chart $\phi_0 : U_0 \to \mathbb{R}^m$ has support in $(0,1)^m = \phi_0(U_0)$ and can be smoothly extended to all of $\mathbb{R}^m$ by setting it equal to zero on $\mathbb{R}^m \setminus (0,1)^m$. Moreover,

$$\int_{\mathbb{R}^m} (\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega) = \int_M \sum_{i=1}^n \psi_i^*(\rho_i\omega) = 0.$$

Hence it follows from Step 1 that there is an $(m-1)$-form $\tau_0 \in \Omega_c^{m-1}(\mathbb{R}^m)$ with support in $(0,1)^m$ such that

$$d\tau_0 = (\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega).$$

Thus $\phi_0^*\tau_0 \in \Omega_c^{m-1}(U_0)$ has compact support in U_0 and therefore extends to all of M by setting it equal to zero on $M \setminus U_0$. This extension satisfies

$$d\phi_0^*\tau_0 = \sum_{i=1}^n \psi_i^*(\rho_i\omega)$$

and hence

$$\omega = \sum_{i=1}^n \psi_i^*(\rho_i\omega) - \sum_{i=1}^n (\psi_i^*(\rho_i\omega) - \rho_i\omega) = d\phi_0^*\tau_0 - \sum_{i=1}^n d\tau_i = d\tau$$

with

$$\tau := \phi_0^*\tau_0 - \sum_{i=1}^n \tau_i \in \Omega_c^{m-1}(M).$$

This proves Step 2 and the theorem. □

Exercise 7.39. Let M be a compact connected oriented smooth m-manifold without boundary and let Λ be a manifold. Let $\Lambda \to \Omega^m(M) : \lambda \mapsto \omega_\lambda$ be a smooth family of m-forms on M such that

$$\int_M \omega_\lambda = 0$$

for every $\lambda \in \Lambda$. Prove that there is a smooth family of $(m-1)$-forms $\Lambda \to \Omega^{m-1}(M) : \lambda \mapsto \tau_\lambda$ such that

$$d\tau_\lambda = \omega_\lambda$$

for every $\lambda \in \Lambda$. **Hint:** Use the argument in the proof of Theorem 7.38 to construct a linear operator

$$h : \left\{ \omega \in \Omega^m(M) \,\Big|\, \int_M \omega = 0 \right\} \to \Omega^{m-1}(M)$$

such that

$$\int_M \omega = 0 \qquad \Longrightarrow \qquad dh\omega = \omega$$

for every $\omega \in \Omega^m(M)$. Find an explicit formula for the operator h. Note that U_i, ρ_i, ψ_i can be chosen once and for all, independent of ω.

Corollary 7.40. *Let M be a compact connected oriented m-manifold without boundary. Then the map*

$$\Omega^m(M) \to \mathbb{R} : \omega \mapsto \int_M \omega$$

induces an ismorphism

$$H^m(M) \cong \mathbb{R}.$$

Proof. The kernel of this map is the space of exact forms, by Theorem 7.38. Hence the induced homomorphism on deRham cohomology is bijective. □

Exercise 7.41. Let M be a compact connected nonorientable m-manifold without boundary. Prove that every m-form on M is exact and hence

$$H^m(M) = 0.$$

Hint: Let $\pi : \widetilde{M} \to M$ be the oriented double cover of M. More precisely, a point in $\widetilde{M}$ is a pair (p, o) consisting of a point $p \in M$ and an orientation o of T_pM. Prove that $\widetilde{M}$ is a compact connected oriented m-dimensional manifold without boundary and that $\pi : \widetilde{M} \to M$ is a local diffeomorphism. Prove that the integral of $\pi^*\omega$ vanishes over $\widetilde{M}$ for every $\omega \in \Omega^m(M)$.

7.5.2 The Degree Theorem

The next theorem relates the integral of a differential form to the integral of its pullback under an arbitrary smooth map between manifolds of the same dimension.

Theorem 7.42 (Degree Theorem). *Let M and N be compact oriented smooth m-manifolds without boundary and suppose that N is connected. Then, for every smooth map $f : M \to N$ and every $\omega \in \Omega^m(N)$, we have*

$$\int_M f^*\omega = \deg(f) \int_N \omega$$

Proof. Let $q \in N$ be a regular value of f. Then $f^{-1}(q)$ is a finite subset of M. Denote the elements of this set by $p_1, \dots, p_n$ and let $\varepsilon_i = \pm 1$ according to whether or not $df(p_i) : T_{p_i}M \to T_qN$ is orientation preserving or orientation reversing. Thus

$$f^{-1}(q) = \{p_1, \dots, p_n\}, \quad \varepsilon_i = \operatorname{sign}\det(df(p_i)), \quad \deg(f) = \sum_{i=1}^n \varepsilon_i. \tag{7.24}$$

Next we observe that there are open neighborhoods $V \subset N$ of q and $U_i \subset M$ of p_i for $i = 1, \dots, n$ satisfying the following conditions.

(a) f restricts to a diffeomorphism from U_i to V for every i; it is orientation preserving when $\varepsilon_i = 1$ and orientation reversing when $\varepsilon_i = -1$.

(b) The sets U_i are pairwise disjoint.

(c) $f^{-1}(V) = U_1 \cup \cdots \cup U_n$.

In fact, since $df(p_i) : T_{p_i}M \to T_qN$ is a vector space isomorphism, it follows from the implicit function theorem that there are connected open neighborhoods U_i of p_i and V_i of q such that $f|_{U_i} : U_i \to V_i$ is a diffeomorphism. Shrinking the sets U_i, if necessary, we may assume $U_i \cap U_j = \emptyset$ for $i \neq j$. Now take

$$V := V_1 \cap \cdots \cap V_n \setminus f(M \setminus (U_1 \cup \cdots \cup U_n))$$

and replace U_i by the set $U_i \cap f^{-1}(V)$. These sets satisfy (a), (b), and (c).

If $\omega \in \Omega^m(N)$ is supported in V then

$$\int_M f^*\omega = \sum_{i=1}^n \int_{U_i} f^*\omega = \sum_{i=1}^n \varepsilon_i \int_V \omega = \deg(f) \int_N \omega.$$

Here the first equation follows from (b) and (c), the second equation follows from (a) and Exercise 7.25, and the last equation follows from (7.24). Now let $\omega \in \Omega^m(N)$ is any m-form and choose $\omega' \in \Omega^m(N)$ such that

$$\mathrm{supp}(\omega') \subset V, \qquad \int_N \omega' = \int_N \omega.$$

Then, by Theorem 7.38, there is a $\tau \in \Omega^{m-1}(N)$ such that

$$d\tau = \omega - \omega'.$$

Hence

$$\begin{aligned} \int_M f^*\omega &= \int_M f^*(\omega' + d\tau) \\ &= \int_M f^*\omega' \\ &= \deg(f) \int_N \omega' \\ &= \deg(f) \int_N \omega. \end{aligned}$$

Here the last but one equality follows from the fact that ω' is supported in V. This proves the theorem. □

7.5.3 The Gauss–Bonnet Formula

Let M be an oriented Riemannian m-manifold. Then there is a unique m-form

$$\mathrm{dvol}_M \in \Omega^m(M)$$

called the **volume form of** M, such that $(\mathrm{dvol}_M)_p(e_1, \dots, e_m) = 1$ for every $p \in M$ and every positively oriented orthonormal basis $e_1, \dots, e_m$ of T_pM.

Exercise 7.43. Let M be an oriented Riemannian m-manifold equipped with an oriented atlas $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ and a metric tensor $g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$. Prove that the volume form dvol_M is in local coordinates given by

$$(\mathrm{dvol}_M)_\alpha = \sqrt{\det(g_\alpha(x))}dx^1 \wedge \cdots dx^m.$$

Let $M \subset \mathbb{R}^{m+1}$ be a compact m-dimensional manifold without boundary. Then M inherits a Riemannian metric from the standard Euclidean inner product on $\mathbb{R}^{m+1}$ and it carries a **Gauss map**

$$\nu : M \to S^m$$

defined as follows. The complement of M in $\mathbb{R}^{m+1}$ has two components, one bounded and one unbounded. These components are distinguished by the mod-2 degree of the map $f_x : M \to S^m$ defined by $f_x(p) := \frac{p-x}{|p-x|}$ for $p \in M$. The bounded component is the set of all $x \in \mathbb{R}^{m+1} \setminus M$ that satisfy $\deg_2(f_x) = 1$ and its closure will be denoted by W. Thus $W \subset \mathbb{R}^{m+1}$ is a compact connected oriented manifold with boundary $\partial W = M$ and we orient M as the boundary of W. The Gauss map $\nu : M \to S^m$ is characterized by the condition that $\nu(p) \in S^m$ is the unique unit vector that is orthogonal to T_pM and points out of W. The volume form $\mathrm{dvol}_M \in \Omega^m(M)$ associated to the metric and orientation of M is then given by the explicit formula

$$(\mathrm{dvol}_M)_p(v_1, \dots, v_m) = \det(\nu(p), v_1, \dots, v_m).$$

Moreover, the derivative of the Gauss map at $p \in M$ is a linear map from T_pM to itself because $T_{\nu(p)}S^m = \nu(p)^\perp = T_pM$. The **Gaussian curvature** of M is the function $K : M \to \mathbb{R}$ defined by

$$K(p) := \det\bigl(d\nu(p) : T_pM \to T_pM\bigr).$$

When M is even dimensional, this function is independent of the choice of the Gauss map. In m is odd then replacing ν by $-\nu$ changes the sign of K.

Theorem 7.44 (Gauss–Bonnet). *Let $M \subset \mathbb{R}^{m+1}$ be a compact m-dimensional submanifold without boundary. Then*

$$\int_M K \mathrm{dvol}_M = \frac{\mathrm{Vol}(S^m)}{2} \chi(M), \tag{7.25}$$

where $\chi(M)$ denotes the Euler characteristic of M.

Remark 7.45. *When $m = 2n$ we have*

$$\frac{\mathrm{Vol}(S^{2n})}{2} = \frac{2^{2n} n!}{(2n)!} \pi^n.$$

When m is odd the Euler characteristic of M is zero.

Proof of Theorem 7.44. The Gauss map of S^m is the identity. Hence the volume form on S^m is given by

$$\mathrm{dvol}_{S^m} = \sum_{i=1}^{m+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{m+1}$$

or, equivalently,

$$(\mathrm{dvol}_{S^m})_x(\xi_1, \ldots, \xi_m) = \det(x, \xi_1, \ldots, \xi_m)$$

for $x \in S^m$ and $\xi_1, \ldots, \xi_m \in T_x S^m = x^\perp$. Hence the pullback of dvol_{S^2} under the Gauss map is given by

$$\begin{aligned}
(\nu^* \mathrm{dvol}_{S^m})_p(v_1, \ldots, v_m) &= (\mathrm{dvol}_{S^m})_{\nu(p)}(d\nu(p)v_1, \ldots, d\nu(p)v_m) \\
&= \det(\nu(p), d\nu(p)v_1, \ldots, d\nu(p)v_m) \\
&= K(p) \det(\nu(p), v_1, \ldots, v_m) \\
&= K(p)(\mathrm{dvol}_M)_p(v_1, \ldots, v_m)
\end{aligned}$$

for $p \in M$ and $v_1, \ldots, v_m \in T_pM = \nu(p)^\perp$. Thus

$$K \mathrm{dvol}_M = \nu^* \mathrm{dvol}_{S^m}.$$

Since the degree of the Gauss map is half the Euler characteristic of M, by the Poincaré–Hopf Theorem, it follows from the Degree Theorem 7.42 that

$$\int_M K \mathrm{dvol}_M = \int_M \nu^* \mathrm{dvol}_{S^m} = \deg(\nu) \int_{S^m} \mathrm{dvol}_{S^m} = \frac{\chi(M)}{2} \mathrm{Vol}(S^m).$$

This proves the theorem. □

Remark 7.46. We shall prove in Section 8.2 that the deRham cohomology of a compact manifold M (with or without boundary) is finite dimensional and in Section 8.4 that the Euler characteristic of a compact oriented m-manifold without boundary is the alternating sum of the **Betti numbers** $b_i := \dim H^i(M)$:

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \dim H^i(M).$$

This formula continues to hold for nonorientable manifolds.

7.5.4 Moser Isotopy

Definition 7.47. *Let M be a smooth m-manifold. A* **volume form** *on M is a nowhere vanishing differential m-form on M. If M is oriented, a volume form $\omega \in \Omega^m(M)$ is called* **compatible with the orientation** *if*

$$\omega_p(v_1, \ldots, v_m) > 0 \tag{7.26}$$

for every $p \in M$ and every positively oriented basis $v_1, \ldots, v_m$ of T_pM. If a volume form ω on an oriented m-manifold M is compatible with the orientation we write $\omega > 0$.

Lemma 7.48. *A manifold M admits a volume form if and only if it is orientable.*

Proof. If $\omega \in \Omega^m(M)$ is a volume form then $\omega_p(v_1, \ldots, v_m) \neq 0$ for every $p \in M$ and every basis $v_1, \ldots, v_m$ of T_pM. Hence a volume form on M determines an orientation of each tangent space T_pM: a basis $v_1, \ldots, v_m$ is called *positively oriented* if (7.26) holds. These orientation fit together smoothly. Namely, fix a point $p_0 \in M$ and a positive basis $v_1, \ldots, v_m$ of $T_{p_0}M$ and choose vector fields $X_1, \ldots, X_m \in \mathrm{Vect}(M)$ such that $X_i(p_0) = v_i$ for $i = 1, \ldots, m$. Then there is a connected open neighborhood $U \subset M$ of p_0 such that the vectors $X_1(p), \ldots X_m(p)$ form a basis of T_pM for every $p \in U$. Hence the function

$$U \to \mathbb{R} : p \mapsto \omega_p(X_1(p), \ldots, X_m(p))$$

is everywhere nonzero and hence is everywhere positive, because it is positive at $p = p_0$. Thus the vectors $X_1(p), \ldots, X_m(p)$ form a positive basis of T_pM for every $p \in U$.

Here is a different argument. Given a volume form $\omega \in \Omega^m(M)$ we can choose an atlas $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ such that the forms

$$\omega_\alpha := (\phi_\alpha)_* \omega \in \Omega^m(\phi_\alpha(U_\alpha))$$

in local coordinates have the form

$$\omega_\alpha = f_\alpha dx^1 \wedge \cdots \wedge dx^m, \qquad f_\alpha > 0.$$

It follows that

$$d(\phi_\beta \circ \phi_\alpha^{-1})(x) = \frac{f_\alpha(x)}{f_\beta(\phi_\beta \circ \phi_\alpha^{-1})(x)} > 0$$

for all α, β and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. Hence the atlas $\{U_\alpha, \phi_\alpha\}_\alpha$ is oriented.

Conversely, suppose M is oriented. Then one can choose a Riemannian metric and take $\omega = \mathrm{dvol}_M$ to be the volume form associated to the metric and orientation. Alternatively, choose an atlas $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ on M such that the transition maps $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are orientation preserving diffeomorphisms for all α and β. Let $\rho_\alpha : M \to [0,1]$ be a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ so that

$$\mathrm{supp}\rho_\alpha \subset U_\alpha, \qquad \sum_\alpha \rho_\alpha \equiv 1.$$

Define $\omega \in \Omega^m(M)$ by

$$\omega := \sum_\alpha \rho_\alpha \phi_\alpha^* dx^1 \wedge \cdots \wedge dx^m,$$

where $\rho_\alpha \phi_\alpha^* dx^1 \wedge \cdots \wedge dx^m \in \Omega_c^m(U_\alpha)$ is extended to all of M by setting it equal to zero on $M \setminus U_\alpha$. Then we have

$$\omega_p(v_1, \dots, v_m) := \sum_{p \in U_\alpha} \rho_\alpha(p) \det(d\phi_\alpha(p)v_1, \dots, d\phi_\alpha(p)v_m)$$

for $p \in M$ and $v_1, \dots, v_m \in T_pM$. Here the sum is understood over all α such that $p \in U_\alpha$. For each $p \in M$ and each basis $v_1, \dots, v_m$ of T_pM all the summands have the same sign and at least one summand is nonzero. Hence ω is a volume form on M and is compatible with the orientation determined by the atlas. □

Theorem 7.49 (**Moser Isotopy**). *Let M be a compact connected oriented m-manifold without boundary and $\omega_0, \omega_1 \in \Omega^m(M)$ be volume forms such that*

$$\int_M \omega_0 = \int_M \omega_1.$$

Then there is a diffeomorphism $\psi : M \to M$, isotopic to the identity, such that $\psi^* \omega_1 = \omega_0$.

Proof. We prove that ω_0 and ω_1 have the same sign on each basis of each tangent space. Let $U \subset M$ be the set of all $p \in M$ such that the real numbers $(\omega_0)_p(v_1, \ldots, v_m)$ and $(\omega_1)_p(v_1, \ldots, v_m)$ have the same sign for some (and hence every) basis $v_1, \ldots, v_m$ of T_pM. Then U and $M \setminus U$ are open sets because ω_0 and ω_1 are volume forms, $U \neq \emptyset$ because the integral of ω_0 and ω_1 agree, and hence $U = M$ because M is connected. Thus ω_0 and ω_1 determine the same orientation of M. Hence the convex combinations

$$\omega_t := (1-t)\omega_0 + t\omega_1, \qquad 0 \le t \le 1,$$

are all volume forms on M. The idea of the proof is to find a smooth isotopy $\psi_t \in \mathrm{Diff}(M)$, $0 \le t \le 1$, starting at the identity, such that

$$\psi_t^*\omega_t = \omega_0 \tag{7.27}$$

for every t. Now every isotopy starting at the identity determines, and is determined by, a smooth family of vector fields $X_t \in \mathrm{Vect}(M)$, $0 \le t \le 1$, via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}. \tag{7.28}$$

By assumption the integral of $\omega_1 - \omega_0$ vanishes over M. Hence, by Theorem 7.38, there is an $(m-1)$-form $\tau \in \Omega^{m-1}(M)$ such that

$$d\tau = \omega_1 - \omega_0 = \partial_t\omega_t.$$

If ψ_t and X_t are related by (7.28) it follows from Cartan's formula in Corollary 7.32 that

$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*(\mathcal{L}_{X_t}\omega_t + \partial_t\omega_t) = \psi_t^* d(\iota(X_t)\omega_t + \tau). \tag{7.29}$$

By Exercise 7.50 below there is a smooth family of vector fields

$$X_t := -I_{\omega_t}^{-1}(\tau) \in \mathrm{Vect}(M), \qquad \iota(X_t)\omega_t + \tau = 0.$$

Let $\psi_t \in \mathrm{Diff}(M)$, $0 \le t \le 1$, be the isotopy of M determined by the vector fields X_t via equation (7.28). Then it follows from (7.29) that the volume form $\psi_t^*\omega_t$ is independent of t and therefore satisfies (7.27). Hence the diffeomorphism $\psi := \psi_1$ satisfies the requirements of the theorem. □

Exercise 7.50. Let M be a smooth m-manifold and $\omega \in \Omega^m(M)$ be a volume form. Prove that the linear map

$$I_\omega : \mathrm{Vect}(M) \to \Omega^{m-1}(M), \qquad I_\omega(X) := \iota(X)\omega,$$

is a vector space isomorphism.

Remark 7.51. Let M be a compact connected oriented smooth m-manifold without boundary. Fix a volume form ω_0 and denote the group of volume preserving diffeomorphisms by

$$\mathrm{Diff}(M, \omega_0) := \{\phi \in \mathrm{Diff}(M) \,|\, \phi^*\omega_0 = \omega_0\}.$$

One can use Moser isotopy to prove that the inclusion of the group of volume preserving diffeomorphisms into the group of all diffeomorphisms is a homotopy equivalence. This is understood with respect to the C^∞-topology on the group of diffeomorphisms. A sequence ψ_ν converges in this topology, by definition, if it converges uniformly with all derivatives.

To prove the assertion consider the set

$$\mathscr{V}(M) := \left\{ \omega \in \Omega^m(M) \,\middle|\, \omega \text{ is a volume form and } \int_M \omega = 1 \right\}$$

of all volume forms on M with volume one and assume $\omega_0 \in \mathscr{V}(M)$. The group $\mathrm{Diff}(M)$ acts on $\mathscr{V}(M)$ and the isotropy subgroup of ω_0 is $\mathrm{Diff}(M, \omega_0)$. Theorem 7.49 asserts that the map

$$\mathrm{Diff}(M) \to \mathscr{V}(M) : \psi \mapsto \psi^*\omega_0$$

is surjective. Moreover, there is a continuous map

$$\mathscr{V}(M) \to \mathrm{Diff}(M) : \omega \mapsto \psi_\omega$$

such that $\psi_\omega^*\omega = \omega_0$ for every $\omega \in \mathscr{V}(M)$ and $\psi_{\omega_0} = \mathrm{id}$. To see this construct an affine map $\mathscr{V}(M) \to \Omega^{m-1}(M) : \omega \mapsto \tau_\omega$ such that $d\tau_\omega = \omega - \omega_0$ for every $\omega \in \mathscr{V}(M)$, following Exercise 7.39, and then use the argument in the proof of Theorem 7.49 to find ψ_ω. It follows that the map

$$\mathrm{Diff}(M) \to \mathscr{V}(M) \times \mathrm{Diff}(M, \omega_0) : \psi \mapsto (\psi^*\omega_0, \psi \circ \psi_{\psi^*\omega_0}) \qquad (7.30)$$

is a homeomophism with inverse $(\omega, \phi) \mapsto \phi \circ \psi_\omega^{-1}$. Since $\mathcal{V}(M)$ is a convex subset of $\Omega^m(M)$ it is contractible and hence the inclusion of $\mathrm{Diff}(M, \omega_0)$ into $\mathrm{Diff}(M)$ is a homotopy equivalence. (See Definitions 8.3 and 8.7 below.)

Exercise 7.52. Prove that there are metrics on $\mathrm{Diff}(M)$ and $\Omega^m(M)$ that induce the C^∞-topology on these spaces. Prove that the map (7.30) is a homeomorphism. **Hint:** If $d : X \times X \to \mathbb{R}$ is a metric so is $d/(1 + d)$.

Chapter 8

De Rham Cohomology

In this chapter we take a closer look at the deRham cohomology groups of a smooth manifold that were introduced at the end of Section 7.2. Here we follow closely the classical textbook of Bott and Tu [2]. An immediate consequence of Cartan's formula in Theorem 7.30 is the observation that smoothly homotopic maps induce the same homomorphism on deRham cohomology, that homotopy equivalent manifolds have the same deRham cohomology, and that the deRham cohomology of a contractible space vanishes in positive degrees. In the case of Euclidean space this is a consequence of the Poincaré Lemma which follows directly from Cartan's formula. These observations are discussed in Section 8.1, which closes with the computation of the deRham cohomology of a sphere. This computation is a special case of the Mayer–Vietoris argument, the subject of Section 8.2. It is a powerful tool in differential and algebraic topology and can be used, for example, to prove that the deRham cohomology groups are finite dimensional and to establish the Künneth formula for the deRham cohomology of a product manifold. Section 8.3 extends the previous discussion to compactly supported deRham cohomology and Section 8.4 is devoted to Poincaré duality, which again can be proved with the Mayer–Vietoris argument. Using Poincaré duality and the Künneth formula one can then show that the Euler characteristic of a compact oriented manifold without boundary, originally defined as the algebraic number of zeroes of a generic vector field, is indeed equal to the alternating sum of the Betti-numbers. A natural generalization of the Mayer–Vietoris sequence is the Čech–deRham complex which will be discussed in Section 8.5. In particular, we show that the deRham cohomology of a manifold is, under suitable hypotheses, isomorphic to the Čech cohomology.

8.1 The Poincaré Lemma

Let M be a smooth m-manifold, N be a smooth n-manifold, and $f : M \to N$ be a smooth map. By Lemma 7.21 the pullback of differential forms under f commutes with the exterior differential:

$$f^* \circ d = d \circ f^*. \tag{8.1}$$

In other words, the following diagram commutes:

$$\begin{array}{ccccccc}
\Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \cdots. \\
\uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \\
\Omega^0(N) & \xrightarrow{d} & \Omega^1(N) & \xrightarrow{d} & \Omega^2(N) & \xrightarrow{d} & \cdots
\end{array}$$

Thus $f^* : \Omega^k(N) \to \Omega^k(M)$ is a linear map which assigns closed forms to closed forms and exact forms to exact forms. Hence it descends to a homomorphism on cohomology, still denoted by f^*:

$$H^k(N) \to H^k(M) : [\omega] \mapsto f^*[\omega] := [f^*\omega].$$

If $g : N \to Q$ is another smooth map between smooth manifolds then, by Lemma 7.14, we have

$$(g \circ f)^* = f^* \circ g^* : H^k(Q) \to H^k(M).$$

Moreover, it follows from Lemmas 7.14 and 7.21 that deRham cohomology is equipped with a **cup product structure**

$$H^k(M) \times H^\ell(M) \to H^{k+\ell}(M) : ([\omega], [\tau]) \mapsto [\omega] \cup [\tau] := [\omega \wedge \tau]$$

and that the cup product is preserved by pullback.

Theorem 8.1. *If $f_0, f_1 : M \to N$ are smoothly homotopic then there is a collection of linear maps $h : \Omega^k(N) \to \Omega^{k-1}(M)$, one for every nonnegative integer k, such that*

$$f_1^* - f_0^* = d \circ h + h \circ d : \Omega^k(N) \to \Omega^k(M) \tag{8.2}$$

for every nonnegative integer k. In particular, the homomorphisms induced by f_0 and f_1 on deRham cohomology agree:

$$f_0^* = f_1^* : H^*(N) \to H^*(M).$$

Proof. Choose a smooth homotopy $F : [0,1] \times M \to N$ satisfying

$$F(0,p) = f_0(p), \qquad F(1,p) = f_1(p)$$

for every $p \in M$, and for $0 \le t \le 1$, define $f_t : M \to N$ by

$$f_t(p) := F(t,p).$$

By Theorem 7.30, we have

$$\frac{d}{dt} f_t^* \omega = dh_t\omega + h_t d\omega$$

for $\omega \in \Omega^k(N)$, where $h_t : \Omega^k(N) \to \Omega^{k-1}(M)$ is defined by

$$(h_t\omega)_p(v_1, \ldots, v_{k-1}) := \omega_{f_t(p)}(\partial_t f_t(p), df_t(p)v_1, \ldots, df_t(p)v_{k-1})$$

for $p \in M$ and $v_i \in T_pM$. Integrating over t we find

$$f_1^*\omega - f_0^*\omega = \int_0^1 \frac{d}{dt} f_t^*\omega \, dt = dh\omega + hd\omega$$

where $h : \Omega^k(N) \to \Omega^{k-1}(M)$ is defined by

$$(h\omega)_p(v_1, \ldots, v_{k-1}) := \int_0^1 \omega_{f_t(p)}(\partial_t f_t(p), df_t(p)v_1, \ldots, df_t(p)v_{k-1}) \, dt \tag{8.3}$$

for $p \in M$ and $v_i \in T_pM$. This proves the theorem. □

Remark 8.2. In homological algebra equation (8.1) says that

$$f^* : \Omega^*(N) \to \Omega^*(M)$$

is a **chain map**. Equation (8.2) says that the chain maps f_0^* and f_1^* are **chain homotopy equivalent** and the map

$$h : \Omega^*(N) \to \Omega^{*-1}(M)$$

is called a **chain homotopy equivalence** from f_0^* to f_1^*. In other words, smoothly homotopic maps between manifold induce chain homotopy equivalent chain maps between the associated deRham cochain complexes. Chain homotopy equivalent chain maps always descend to the same homorphism on (co)homology.

Definition 8.3. *Two manifolds M and N are called* **homotopy equivalent** *if there exist smooth maps $f : M \to N$ and $g : N \to M$ such that the compositions*

$$g \circ f : M \to M, \qquad f \circ g : N \to N$$

are both homotopic to the respective identity maps. If this holds the maps f and g are called **homotopy equivalences** *and g is called a* **homotopy inverse** *of f.*

Exercise 8.4. The closed unit disc in $\mathbb{R}^m$ (an m-manifold with boundary) is homotopy equivalent to a point (a 0-manifold without boundary).

Corollary 8.5. *Homotopy equivalent manifolds have isomorphic deRham cohomology (including the product structures).*

Proof. Let $f : M \to N$ be a homotopy equivalence and $g : N \to M$ be a homotopy inverse of f. Then it follows from Theorem 8.1 that

$$f^* \circ g^* = (g \circ f)^* = \mathrm{id} : H^*(M) \to H^*(M)$$

and

$$g^* \circ f^* = (f \circ g)^* = \mathrm{id} : H^*(N) \to H^*(N).$$

Hence $f^* : H^*(N) \to H^*(M)$ is a vector space isomorphism and

$$(f^*)^{-1} = g^* : H^*(M) \to H^*(N).$$

This proves the corollary. □

Example 8.6. For every smooth manifold M we have

$$H^*(M) \cong H^*(\mathbb{R} \times M).$$

To see this, define $\pi : \mathbb{R} \times M \to M$ and $\iota : M \to \mathbb{R} \times M$ by

$$\pi(s, p) := p, \qquad \iota(p) := (0, p)$$

for $s \in \mathbb{R}$ and $p \in M$. Then $\pi \circ \iota = \mathrm{id} : M \to M$ and $\iota \circ \pi : \mathbb{R} \times M \to \mathbb{R} \times M$ is homotopic to the identity. An explicit homotopy is given by

$$f_t : \mathbb{R} \times M \to \mathbb{R} \times M, \qquad f_t(s, p) := (st, p), \qquad f_0 = \iota \circ \pi, \qquad f_1 = \mathrm{id}.$$

Hence M and $\mathbb{R} \times M$ are homotopy equivalent and so the assertion follows from Corollary 8.5. Explicitly, $\pi^* : H^*(M) \to H^*(\mathbb{R} \times M)$ is an isomorphism with inverse $\iota^* : H^*(\mathbb{R} \times M) \to H^*(M)$.

Definition 8.7. *A smooth manifold M is called* **contractible** *if the identity map on M is homotopic to a constant map.*

Exercise 8.8. Every contractible manifold is nonempty and connected.

Exercise 8.9. A manifold is contractible if and only if it is homotopy equivalent to a point.

Exercise 8.10. Every nonempty geodesically convex open subset of a Riemannian m-manifold without boundary is contractible.

Corollary 8.11 (**Poincaré Lemma**). *Let M be a contractible manifold. Then there is a collection of linear maps $h : \Omega^k(M) \to \Omega^{k-1}(M)$, one for every nonnegative integer k, such that*

$$d \circ h + h \circ d = \mathrm{id} : \Omega^k(M) \to \Omega^k(M), \qquad k \geq 1. \tag{8.4}$$

Hence $H^0(M) = \mathbb{R}$ and $H^k(M) = 0$ for $k \geq 1$.

Proof. Let $p_0 \in M$ and $[0,1] \times M \to M : (t,p) \mapsto f_t(p)$ be a smooth homotopy such that $f_0(p) = p_0$ and $f_1(p) = p$ for every $p \in M$. Let $h : \Omega^k(M) \to \Omega^{k-1}(M)$ be given by (8.3). Then, for every k-form $\omega \in \Omega^k(M)$ with $k \geq 1$, it follows from Theorem 8.1 that

$$\omega = f_1^*\omega - f_0^*\omega = dh\omega - hd\omega.$$

(The assumption $k \geq 1$ is needed in the first equation.) Hence, for $k \geq 1$, every closed k-form on M is exact and so $H^k(M) \cong 0$. Since M is connected we have $H^0(M) = \mathbb{R}$. This proves the corollary. □

Example 8.12. The Euclidean space $\mathbb{R}^m$ is contractible. An explicit homotopy from a constant map to the identity is given by $f_t(x) := tx$ for $0 \leq t \leq 1$ and $x \in \mathbb{R}^m$. Hence

$$H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R}, & \text{for } k = 0, \\ 0, & \text{for } k \geq 1. \end{cases}$$

The chain homotopy equivalence $h : \Omega^k(\mathbb{R}^m) \to \Omega^{k-1}(\mathbb{R}^m)$ associated to the above homotopy f_t via (8.3) is given by

$$(h\omega)(x;\xi_1,\dots,\xi_{k-1}) = \int_0^1 t^{k-1}\omega(x;tx,\xi_1,\dots,\xi_{k-1})\,dt \tag{8.5}$$

for $\omega \in \Omega^k(\mathbb{R}^m)$ and $x, \xi_1, \dots, \xi_{k-1} \in \mathbb{R}^m$. By Corollary 8.5 it satisfies

$$d \circ h + h \circ d = \mathrm{id} : \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^m)$$

for $k \geq 1$. This is the **Poincaré Lemma** in its original form.

Example 8.13. For $m \geq 1$ the deRham cohomology of the unit sphere

$$S^m \subset \mathbb{R}^{m+1}$$

is given by

$$H^k(S^m) = \begin{cases} \mathbb{R}^m, & \text{for } k = 0 \text{ and } k = m, \\ 0, & \text{for } 1 \leq k \leq m-1. \end{cases}$$

That $H^0(S^m) = \mathbb{R}$ follows from Example 7.22 because S^m is connected (whenever $m \geq 1$). That $H^m(S^m) = \mathbb{R}$ follows from Corollary 7.40 because S^m is a compact connected oriented manifold without boundary.

We prove that

$$H^1(S^m) = 0$$

for every $m \geq 2$. To see this consider the open sets

$$U^\pm := S^m \setminus \{(0, \dots, 0, \mp 1)\}.$$

Their union is S^m, each set U^+ and U^- is diffeomorphic to $\mathbb{R}^m$ via stereographic projection, and their intersection $U^+ \cap U^-$ is diffeomorphic to $\mathbb{R}^m \setminus \{0\}$ and hence to $\mathbb{R} \times S^{m-1}$:

$$U^+ \cong U^- \cong \mathbb{R}^m, \qquad U^+ \cap U^- \cong \mathbb{R} \times S^{m-1}.$$

In particular, the intersection $U^+ \cap U^-$ is connected because $m \geq 2$. Now let $\alpha \in \Omega^1(S^m)$ be a closed 1-form. Then it follows from Example 8.12 that the restrictions of α to U^+ and U^- are exact. Hence there are smooth functions $f^\pm : U^\pm \to \mathbb{R}$ such that

$$\alpha|_{U^+} = df^+, \qquad \alpha|_{U^-} = df^-.$$

The differential of the difference $f^+ - f^- : U^+ \cap U^- \to \mathbb{R}$ vanishes. Since $U^+ \cap U^-$ is connected there is a constant $c \in \mathbb{R}$ such that

$$f^+(x) - f^-(x) = c \qquad \forall\, x \in U^+ \cap U^-.$$

Define $f : S^m \to \mathbb{R}$ by

$$f(x) := \begin{cases} f^-(x) + c, & \text{for } x \in U^-, \\ f^+(x), & \text{for } x \in U^+. \end{cases}$$

This function is well defined and smooth and satisfies $df = \alpha$. Thus we have proved that every closed 1-form on S^m is exact, when $m \geq 2$, and thus $H^1(S^m) = 0$, as claimed.

We prove by induction on m that $H^k(S^m) = 0$ for $1 \le k \le m-1$ and $m \ge 2$. We have just seen that this holds for $m = 2$. Thus let $m \ge 3$ and assume, by induction, that the assertion holds for $m - 1$. We have already shown that $H^1(S^m) = 0$. Thus we fix an integer

$$2 \le k \le m-1$$

and prove that

$$H^k(S^m) = 0.$$

Let $\omega \in \Omega^k(S^m)$ be a closed k-form. By Example 8.12, the restrictions of ω to U^+ and U^- are both exact. Hence there are smooth $(k-1)$-forms $\tau^\pm \in \Omega^{k-1}(U^\pm)$ such that

$$\alpha|_{U^+} = d\tau^+, \qquad \alpha|_{U^-} = d\tau^-.$$

Hence the $(k-1)$-form

$$\tau^+|_{U^+\cap U^-} - \tau^-|_{U^+\cap U^-} \in \Omega^{k-1}(U^+ \cap U^-)$$

is closed. By Example 8.6 and the induction hypothesis, we have

$$H^{k-1}(U^+ \cap U^-) \cong H^{k-1}(\mathbb{R} \times S^{m-1}) \cong H^{k-1}(S^{m-1}) = 0.$$

Hence there is a $(k-2)$-form $\beta \in \Omega^{k-2}(U^+ \cap U^-)$ such that

$$d\beta = \tau^+|_{U^+\cap U^-} - \tau^-|_{U^+\cap U^-}.$$

Now choose a smooth cutoff function $\rho : S^m \to [0,1]$ such that

$$\rho(x) = \begin{cases} 0, & \text{for } x \text{ near } (0,\dots,0,-1), \\ 1, & \text{for } x \text{ near } (0,\dots,0,1), \end{cases}$$

and define $\tau \in \Omega^{k-1}(S^m)$ by

$$\tau := \begin{cases} \tau^- + d(\rho\beta) & \text{on } U^-, \\ \tau^+ - d((1-\rho)\beta) & \text{on } U^+. \end{cases}$$

Then $d\tau = \omega$. Thus we have proved that every closed k-form on S^m is exact and hence $H^k(S^m) = 0$, as claimed.

The computation of the deRham cohomology of S^m in Example 8.13 is an archetypal example of a Mayer–Vietoris argument. More generally, if we have a cover of a manifold by two well chosen open sets U and V, the computation of the deRham cohomology of M can be reduced to the computation of the deRham cohomology of the manifolds U, V, and $U \cap V$ by means of the Mayer–Vietoris sequence. We shall see that this exact sequence is a powerful tool for understanding deRham cohomology.

8.2 The Mayer–Vietoris Sequence

8.2.1 The Short Exact Sequence

Let M be a smooth m-manifold (not necessarily compact or connected and with or without boundary). Let $U, V \subset M$ be open sets such that

$$M = U \cup V.$$

The **Mayer–Vietoris sequence** associated to this open cover by two sets is the sequence of homomorphisms

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j^*} \Omega^k(U \cap V) \longrightarrow 0, \qquad (8.6)$$

where $i^* : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V)$ and $j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ are defined by

$$i^*\omega := (\omega|_U, \omega|_V), \qquad j^*(\omega_U, \omega_V) := \omega_V|_{U\cap V} - \omega_U|_{U\cap V}$$

for $\omega \in \Omega^k(M)$ and $\omega_U \in \Omega^k(U)$, $\omega_V \in \Omega^k(V)$. Thus i^* is given by restriction and j^* by restriction followed by subtraction.

Lemma 8.14. *The Mayer–Vietoris sequence* (8.6) *is exact.*

Proof. That i^* is injective, is obvious: if $\omega \in \Omega^k(M)$ vanishes on U and on V then it vanishes on all of M. That the image of i^* agrees with the kernel of j^* is also obvious: if $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ agree on the intersection $U \cap V$, then they determine a unique global k-form $\omega \in \Omega^k(M)$ such that $\omega|_U = \omega_U$ and $\omega|_V = \omega_V$.

We prove that j^* is surjective. Choose a partition of unity subordinate to the open cover $M = U \cup V$. It consists of two smooth functions $\rho_U : M \to [0,1]$ and $\rho_V : M \to [0,1]$ satisfying

$$\mathrm{supp}(\rho_U) \subset U, \qquad \mathrm{supp}(\rho_V) \subset V, \qquad \rho_U + \rho_V \equiv 1.$$

Now let $\omega \in \Omega^k(U \cap V)$ and define $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ by

$$\omega_U := \begin{cases} -\rho_V\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V, \end{cases} \qquad \omega_V := \begin{cases} \rho_U\omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U. \end{cases}$$

Then

$$j^*(\omega_U, \omega_V) = \omega_V|_{U\cap V} - \omega_U|_{U\cap V} = \rho_U\omega + \rho_V\omega = \omega$$

as claimed. This proves the lemma. □

8.2.2 The Long Exact Sequence

The Mayer–Vietoris sequence (8.6) is an example of what is called a **short exact sequence** in homological algebra in that it is short (five terms starting and ending with zero), it is exact, and it consists of chain homomorphisms. Thus the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{k+1}(M) & \xrightarrow{i^*} & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & \xrightarrow{j^*} & \Omega^{k+1}(U\cap V) & \longrightarrow & 0\,. \\ & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\ 0 & \longrightarrow & \Omega^{k}(M) & \xrightarrow{i^*} & \Omega^{k}(U)\oplus\Omega^{k}(V) & \xrightarrow{j^*} & \Omega^{k}(U\cap V) & \longrightarrow & 0 \end{array}$$

Any such short exact sequence gives rise to a **long exact sequence** in cohomology. The relevant boundary operator will be denoted by

$$d^* : H^k(U\cap V) \to H^{k+1}(M)$$

and it is defined as follows. Given a closed form $\omega \in \Omega^k(U\cap V)$ choose a pair $(\omega_U, \omega_V) \in \Omega^k(U)\oplus\Omega^k(V)$ whose image under j^* is ω; then the pair $(d\omega_U, d\omega_V)$ belongs to the kernel of j^*, because ω is closed, and hence belongs to the image of i^*, by exactness; hence there is a unique $(k+1)$-form $d^*\omega \in \Omega^{k+1}(M)$ whose image under i^* is the given pair $(d\omega_U, d\omega_V)$. Since $i^*d(d^*\omega) = di^*(d^*\omega) = d(d\omega_U, d\omega_V) = 0$ and i^* is injective it follows that $d^*\omega$ is closed. Moreover, one can check that the cohomology class of $d^*\omega$ is independent of the choice of the pair (ω_U, ω_V) used in this construction.

In the present setting we have an explicit formula for the operator d^* coming from the proof of Lemma 8.14. Namely, we define an operator

$$d^* : \Omega^k(U\cap V) \to \Omega^{k+1}(M)$$

by

$$d^*\omega := \begin{cases} d\rho_U \wedge \omega & \text{on } U\cap V, \\ 0 & \text{on } M\setminus(U\cap V). \end{cases} \tag{8.7}$$

This operator is well defined because the 1-form $d\rho_U = -d\rho_V$ is supported in $U\cap V$. Moreover, we have

$$d\circ d^* + d^*\circ d = 0$$

and hence d^* assigns closed forms to closed forms and exact forms to exact forms. Thus d^* descends to a homomorphism on cohomology and the reader may check that this is precisely the homomorphism defined by *diagram chasing* as above.

The homomorphisms on de Rham cohomology induced by i^*, j^*, d^* give rise to a long exact sequence

$$\cdots H^k(M) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \xrightarrow{d^*} H^{k+1}(M) \cdots \tag{8.8}$$

which is also called the **Mayer–Vietoris sequence**.

Theorem 8.15. *The Mayer–Vietoris sequence* (8.8) *is exact.*

Proof. That the composition of any two successive homomorphisms is zero follows directly from the definitions.

We prove that $\ker d^* = \operatorname{im} j^*$. Let $\omega \in \Omega^k(U \cap V)$ be a closed k-form such that $d^*[\omega] = [d^*\omega] = 0$. Then the k-form $d^*\omega \in \Omega^{k+1}(M)$ is exact. Thus there is a k-form $\tau \in \Omega^k(M)$ such that $d\tau = d^*\omega$ or, equivalently,

$$d\tau|_{U\cap V} = d\rho_U \wedge \omega, \qquad d\tau|_{M\setminus(U\cap V)} = 0.$$

Define $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ by

$$\omega_U := -\rho_V\omega - \tau|_U, \qquad \omega_V := \rho_U\omega - \tau|_V.$$

Here it is understood that the form $-\rho_V\omega$ on $U\cap V$ is extended to all of U by setting it equal to sero on $U\setminus V$ and the form $\rho_U\omega$ on $U\cap V$ is extended to all of V by setting it equal to zero on $V \setminus U$. The k-forms ω_U and ω_V are closed and hence determine cohomology classes $[\omega_U] \in H^k(U)$ and $[\omega_V] \in H^k(V)$. Moreover

$$\omega_V|_{U\cap V} - \omega_U|_{U\cap V} = \rho_U\omega + \rho_V\omega = \omega$$

and hence $j^*([\omega_U], [\omega_V]) = [\omega]$. Thus we have proved that $\ker d^* = \operatorname{im} j^*$.

We prove that $\ker j^* = \operatorname{im} i^*$. Let $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ be closed k-forms such that $j_*([\omega_U][\omega_V]) = 0$. Then there is a $(k-1)$-form $\tau \in \Omega^{k-1}(U \cap V)$ such that

$$\omega_V|_{U\cap V} - \omega_U|_{U\cap V} = d\tau.$$

By Lemma 8.14 there exist $(k-1)$-forms $\tau_V \in \Omega^{k-1}(U)$ and $\tau_V\Omega^{k-1}(V)$ such that

$$\tau_V|_{U\cap V} - \tau_U|_{U\cap V} = \tau.$$

Combining the last two equations we find that $\omega_U - d\tau_U$ agrees with $\omega_V - d\tau_V$ on $U \cap V$. Hence there is a global k-form $\omega \in \Omega^k(M)$ such that

$$\omega|_U = \omega_U - d\tau_U, \qquad \omega|_V = \omega_V - d\tau_V.$$

This form is obviously closed, its restriction to U is cohomologous to ω_U, and its restriction to V is cohomologous to ω_V. Hence $i^*[\omega] = ([\omega_U], [\omega_V])$. Thus we have proved that $\ker j^* = \operatorname{im} i^*$.

We prove that $\ker i^* = \operatorname{im} d^*$. Let $\omega \in \Omega^k(M)$ be a closed k-form such that $i^*[\omega] = 0$. Then $\omega|_U$ and $\omega|_V$ are exact. Thus there are $(k-1)$-forms $\tau_U \in \Omega^{k-1}(U)$ and $\tau_V \in \Omega^{k-1}(V)$ such that

$$d\tau_U = \omega|_U, \qquad d\tau_V = \omega|_V.$$

Hence the $(k-1)$-form

$$\tau := \tau_V|_{U\cap V} - \tau_U|_{U\cap V} \in \Omega^{k-1}(U \cap V)$$

is closed. We prove that $d^*[\tau] = [\omega]$. To see this, define $\sigma \in \Omega^{k-1}(M)$ by

$$\sigma := \begin{cases} \rho_U\tau_U + \rho_V\tau_V & \text{on } U \cap V, \\ \rho_U\tau_U & \text{on } U \setminus V, \\ \rho_V\tau_V & \text{on } V \setminus U. \end{cases}$$

Then

$$\tau_U = -\rho_V\tau + \sigma|_U, \qquad \tau_V = \rho_U\tau + \sigma|_V.$$

Here the form $\rho_V\tau$ on $U \cap V$ is again understood to be extended to all of U by setting it equal to zero on $U \cap V$ and the form $\rho_U\tau$ on $U \cap V$ is understood to be extended to all of V by setting it equal to zero on $V \setminus U$. Since τ is closed we obtain

$$d^*\tau = \left\{ \begin{array}{ll} -d(\rho_V\tau) & \text{on } U \\ d(\rho_U\tau) & \text{on } V \end{array} \right\} = \left\{ \begin{array}{ll} d\tau_U - d\sigma|_U & \text{on } U \\ d\tau_V - d\sigma|_V & \text{on } V \end{array} \right\} = \omega - d\sigma.$$

Hence $d^*[\tau] = [\omega]$ as claimed. Thus we have proved that $\ker i^* = \operatorname{im} d^*$ and this completes the proof of the theorem. □

Corollary 8.16. *If $M = U \cup V$ is the union of two open sets such that the deRham cohomology of U, V, $U \cap V$ is finite dimensional, then so is the deRham cohomology of M.*

Proof. By Theorem 8.15 the vector space $H^k(M)$ is isomorphic to the direct sum of the image of

$$d^* : H^{k-1}(U \cap V) \to H^k(M)$$

and the image of

$$i^* : H^k(M) \to H^k(U) \oplus H^k(V).$$

As both summands are finite dimensional so is $H^k(M)$. This proves the corollary. □

8.2.3 Finite Good Covers

The previous result can be used to prove finite dimensionality of the deRham cohomology for a large class of manifolds. A collection $\mathscr{U} = \{U_i\}_{i\in I}$ of nonempty open subsets $U_i \subset M$ is called a **good cover** if $M = \bigcup_{i\in I} U_i$ and each intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^m$. $\mathscr{U}$ is called a **finite good cover** if it is a good cover and I is a finite set. Note that the existence of a good cover implies that M has no boundary.

Exercise 8.17. Prove that every compact m-manifold without boundary has a finite good cover. **Hint:** Choose a Riemannian metric and cover M by finitely many geodesic balls of radius at most half the injectivity radius. Show that the intersections are all geodesically convex and use Exercise 8.18.

Exercise 8.18. Prove that every nonempty geodesically convex open subset of a Riemannian m-manifold M without boundary is diffeomorphic to $\mathbb{R}^m$. **Hint 1:** Prove that it is diffeomorphic to a bounded **star shaped** open set $U \subset \mathbb{R}^m$ centered at the origin, so that if $x \in U$ then $tx \in U$ for $0 \le t \le 1$. **Hint 2:** Prove that there is a smooth function $g : \mathbb{R}^m \to \mathbb{R}$ such that $g(x) > 0$ for every $x \in U$, $g(x) = 1$ for $|x|$ sufficiently small, and $g(x) = 0$ for $x \in \mathbb{R}^n \setminus U$. Define $h : U \to [0, \infty)$ by

$$h(x) := \int_0^1 \frac{dt}{g(tx)}.$$

Prove that the map $\phi : U \to \mathbb{R}^m$, $\phi(x) := h(x)x$, is a diffeomorphism. **Hint 3:** There is a **lower semicontinuous** function $f : S^{m-1} \to (0, \infty]$ such that $U = U_f := \left\{rx \,|\, x \in S^{m-1},\ 0 \le r < f(x)\right\}$. (Lower semicontinuity is characterized by the fact that the set U_f is open.) The **Moreau envelopes** of f are the functions

$$(e_n f)(x) := \inf_{y\in S^{m-1}} \left(f(y) + \frac{n}{2} |x - y|^2 \right).$$

They are continuous and real valued (unless $f \equiv \infty$) and they approximate f pointwise from below. Use this to prove that there is a sequence of smooth functions $f_n : S^{m-1} \to \mathbb{R}$ satisfying $0 < f_n < f_{n+1} < f$ for every n and $\lim_{n\to\infty} f_n(x) = f(x)$ for every x. Construct a diffeomorphism from $\mathbb{R}^m$ to U_f that maps the open ball of radius n diffeomorphically onto the set U_{f_n}.

Exercise 8.19. Let M be a compact manifold with boundary. Prove that $M \setminus \partial M$ has a good cover. **Hint:** Choose a Riemannian metric on M that restricts to a product metric in a tubular neighborhood of the boundary.

Corollary 8.20. *If M admits a finite good cover then its deRham cohomology is finite dimensional.*

Proof. The proof is by induction on the number of elements in the good cover. If M has a good cover consisting of precisely one open set then M is diffeomorphic to $\mathbb{R}^m$ and hence its deRham cohomology is one-dimensional by Example 8.12. Now fix an integer $n \geq 2$ and suppose, by induction, that every smooth manifold that admits a good cover by at most $n-1$ open sets has finite dimensional deRham cohomology. Let $M = U_1 \cup U_2 \cup \cdots \cup U_n$ be a good cover and denote

$$U := U_1 \cup \cdots \cup U_{n-1}, \qquad V := U_n.$$

Then $U \cap V$ has a good cover consisting of the open sets $U_i \cap U_n$ for $i = 1, \ldots, n-1$. Hence it follows from the induction hypothesis that the manifolds U, V, $U \cap V$ have finite dimensional deRham cohomology. Thus, by Corollary 8.16, the deRham cohomology of M is finite dimensional as well. This proves the corollary. □

Corollary 8.21. *Every compact manifold M has finite dimensional deRham cohomology.*

Proof. The manifold $M \setminus \partial M$ has a finite good cover and is homotopy equivalent to M. (Prove this!) Hence the assertion follows from Corollary 8.5 and Corollary 8.20. □

8.2.4 The Künneth Formula

Let M and N be smooth manifolds and consider the projections

$$\begin{array}{ccc} M \times N & \xrightarrow{\pi_N} & N \\ \big\downarrow{\scriptstyle \pi_M} & & \\ M & & \end{array}$$

They induce a linear map

$$\Omega^k(M) \otimes \Omega^\ell(N) \to \Omega^{k+\ell}(M \times N) : \omega \otimes \tau \mapsto \pi_M^*\omega \wedge \pi_N^*\tau. \tag{8.9}$$

If ω and τ are closed then so is $\pi_M^*\omega \wedge \pi_N^*\tau$ and if, in addition, one of the forms is exact so is $\pi_M^*\omega \wedge \pi_N^*\tau$. Hence the map (8.9) induces a homomorphism on deRham cohomology

$$\kappa : H^*(M) \otimes H^*(N) \to H^*(M \times N), \quad \kappa([\omega] \otimes [\tau]) := [\pi_M^*\omega \wedge \pi_N^*\tau]. \tag{8.10}$$

Theorem 8.22 (Künneth formula). *If M and N have finite good covers then κ is an isomorphism; thus*

$$H^\ell(M\times N)\cong\bigoplus_{k=0}^{\ell}H^k(M)\otimes H^{\ell-k}(N)$$

for every integer $\ell\geq 0$ and $\dim H^*(M\times N)=\dim H^*(M)\cdot\dim H^*(N)$.

Proof. The proof is by induction on the number n of elements in a good cover of M. If $n=1$ then M is diffeomorphic to $\mathbb{R}^m$. In this case it follows from Example 8.6 that the projection $\pi_N:M\times N\to N$ induces an isomorphism $\pi_N^*:H^*(N)\to H^*(M\times N)$ on deRham cohomology. Moreover, by Example 8.12, we have $H^*(\mathbb{R}^m)=H^0(\mathbb{R}^m)=\mathbb{R}$ and hence κ is an isomorphism, as claimed.

Now fix an integer $n\geq 2$ and assume, by induction, that the Küenneth formula holds for $M\times N$ whenever M admits a good cover by at most $n-1$ open sets. Suppose that $M=U_1\cup U_2\cup\cdots\cup U_n$ is a good cover and denote

$$U:=U_1\cup\cdots\cup U_{n-1},\qquad V:=U_n.$$

Then, by the induction hypothesis, the Künneth formula holds for $U\times N$, $V\times N$, and $(U\cap V)\times N$. We abbreviate

$$\widetilde{H}^\ell(M):=\bigoplus_{k=0}^{\ell}H^k(M)\otimes H^{\ell-k}(N),\qquad \widehat{H}^\ell(M):=H^\ell(M\times N),$$

so that κ is a homomorphism from $\widetilde{H}^\ell(M)$ to $\widehat{H}^\ell(M)$. Then the Mayer–Vietoris sequence gives rise to the following commutative diagram:

$$\begin{array}{ccccccc}
\widetilde{H}^\ell(M) & \xrightarrow{i^*} & \widetilde{H}^\ell(U)\oplus\widetilde{H}^\ell(V) & \xrightarrow{j^*} & \widetilde{H}^\ell(U\cap V) & \xrightarrow{d^*} & \widetilde{H}^{\ell+1}(M) \\
\downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\
\widehat{H}^\ell(M) & \xrightarrow{i^*} & \widehat{H}^\ell(U)\oplus\widehat{H}^\ell(V) & \xrightarrow{j^*} & \widehat{H}^\ell(U\cap V) & \xrightarrow{d^*} & \widehat{H}^{\ell+1}(M)
\end{array}.$$

That the first two squares in this diagram commute is obvious from the definitions. We examine the third square. It has the form

$$\begin{array}{ccc}
\bigoplus_{k=0}^{\ell}H^k(U\cap V)\otimes H^{\ell-k}(N) & \xrightarrow{d^*} & \bigoplus_{k=0}^{\ell}H^{k+1}(M)\otimes H^{\ell-k}(N) \\
\downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\
H^\ell((U\cap V)\times N) & \xrightarrow{d^*} & H^{\ell+1}(M\times N)
\end{array}.$$

If $\omega \in \Omega^k(U \cap V)$ and $\tau \in \Omega^{\ell-k}(N)$ are closed forms we have

$$\begin{aligned} \kappa d^*(\omega \otimes \tau) &= \pi_M^* d^* \omega \wedge \pi_N^* \tau \\ d^* \kappa(\omega \otimes \tau) &= d^*(\pi_M^* \omega \wedge \pi_N^* \tau). \end{aligned}$$

Recall that $d^*\omega \in \Omega^{k+1}(M)$ is given by $d\rho_U \wedge \omega$ on $U \cap V$ and vanishes on $M \setminus (U \cap V)$, where $\rho_U, \rho_V : M \to [0,1]$ are as in the proof of Lemma 8.14. They also give rise to a partition of unity on $M \times N$, subordinate to the cover by the open sets $U \times N$ and $V \times N$, and defined by

$$\begin{aligned} \pi_M^* \rho_U &= \rho_U \circ \pi_M : M \times N \to [0,1], \\ \pi_M^* \rho_V &= \rho_V \circ \pi_M : M \times N \to [0,1]. \end{aligned}$$

Using this partition of unity for the definition of the boundary operator

$$d^* : \Omega^\ell((U \cap V) \times N) \to \Omega^{\ell+1}(M \times N)$$

in the Mayer–Vietoris sequence for $M \times N$, we obtain the equation

$$\begin{aligned} d^* \kappa(\omega \otimes \tau) &= d^*(\pi_M^* \omega \wedge \pi_N^* \tau) \\ &= d(\pi_M^* \rho_U) \wedge \pi_M^* \omega \wedge \pi_N^* \tau \\ &= \pi_M^*(d\rho_U \wedge \omega) \wedge \pi_N^* \tau \\ &= \pi_M^* d^* \omega \wedge \pi_N^* \tau \\ &= \kappa d^*(\omega \otimes \tau). \end{aligned}$$

on $(U \cap V) \times N$. Since both sides of this equation vanish on $(M \setminus (U \cap V)) \times N$, we have proved that

$$d^* \circ \kappa = \kappa \circ d^*.$$

Thus $\kappa : \widetilde{H}^* \to \widehat{H}^*$ induces a commuting diagram of the Mayer–Vietoris sequences for $\widetilde{H}^*$ and $\widehat{H}^*$. The induction hypothesis asserts that κ is an isomorphism for each of the manifolds U, V, and $U \cap V$. Hence it follows from the Five Lemma 8.23 below that it also is an isomorphism for M. This completes the induction argument and the proof of the Künneth formula. □

Lemma 8.23 (Five Lemma). *Let*

$$\begin{array}{ccccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \xrightarrow{f_1'} & B' & \xrightarrow{f_2'} & C' & \xrightarrow{f_3'} & D' & \xrightarrow{f_4'} & E' \end{array}$$

be a commutative diagram of homomorphisms of abelian groups such that the horizontal sequences are exact. If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms then so is γ.

Proof. Exercise. □

8.3 Compactly Supported Differential Forms

8.3.1 Definition and Basic Properties

Let M be an m-dimensional smooth manifold (possibly with boundary) and, for every integer $k \geq 0$, denote by $\Omega_c^k(M)$ the space of compactly supported k-forms on M. (See Section 7.2.1.) Consider the cochain complex

$$\Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_c^m(M).$$

The cohomology of this complex is called the **compactly supported de-Rham cohomology** of M and will be denoted by

$$H_c^k(M) := \frac{\ker d : \Omega_c^k(M) \to \Omega_c^{k+1}(M)}{\operatorname{im} d : \Omega_c^{k-1}(M) \to \Omega_c^k(M)}$$

for $k = 0, 1, \ldots, m$.

Remark 8.24. If M is compact then every differential form on M has compact support and hence $\Omega_c^*(M) = \Omega^*(M)$ and $H_c^*(M) = H^*(M)$.

Remark 8.25. The compactly supported deRham cohomology of a manifold is not functorial. If $f : M \to N$ is a smooth map (between noncompact manifolds) and $\omega \in \Omega_c^k(N)$ is a compactly supported differential form on N then

$$\operatorname{supp}(f^*\omega) \subset f^{-1}(\operatorname{supp}(\omega)).$$

Thus $f^*\omega$ may not have compact support.

Remark 8.26. If $f : M \to N$ is **proper** in the sense that

$$K \subset N \text{ is compact} \qquad \Longrightarrow \qquad f^{-1}(K) \subset M \text{ is compact},$$

then pullback under f is a cochain map

$$f^* : \Omega_c^*(N) \to \Omega_c^*(M)$$

and thus induces a homomorphism on compactly supported deRham cohomology. By Corollary 7.31 the induced map on cohomology is invariant under proper homotopies. Here it is not enough to assume that each map f_t in a homotopy is proper; one needs the condition that the homotopy $[0,1] \times M \to N : (t,p) \mapsto f_t(p)$ itself is proper.

Remark 8.27. If $\iota : U \to M$ is the inclusion of an open set then every compactly supported differential form on U can be extended to a smooth differential form on all of M by setting it equal to zero on $M \setminus U$. Thus there is an inclusion induced cochain map

$$\iota_* : \Omega_c^*(U) \to \Omega_c^*(M)$$

and a homomorphism on compactly supported deRham cohomology.

These remarks show that the compactly supported deRham cohomology of a noncompact manifold behaves rather differently from the usual deRham cohomology. This is also illustrated by the following examples.

Example 8.28. The compactly supported deRham cohomology of the 1-manifold $M = \mathbb{R}$ is given by

$$H_c^0(\mathbb{R}) = 0, \qquad H_c^1(\mathbb{R}) = \mathbb{R}.$$

That $H_c^0(\mathbb{R}) = 0$ follows from the fact that every compactly supported function $f : \mathbb{R} \to \mathbb{R}$ with $df = 0$ vanishes identically. To prove $H_c^1(\mathbb{R}) = \mathbb{R}$ we observe that a 1-form $\omega \in \Omega_c^1(\mathbb{R})$ can be written in the form

$$\omega = g(x)\,dx,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. Thus $\omega = df$ where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := \int_{-\infty}^x g(t)\,dt$. This function has compact support if and only if the integral of g over $\mathbb{R}$ vanishes. Thus ω belongs to the image of the operator $d : \Omega_c^0(\mathbb{R}) \to \Omega_c^1(\mathbb{R})$ if and only if its integral is zero. This is a special case of Theorem 7.38.

Example 8.29. If M is connected and not compact then every compactly supported locally constant function on M vanishes and hence

$$H_c^0(M) = 0.$$

Example 8.30. If M is a nonempty connected oriented smooth m-dimensionnal manifold without boundary then

$$H_c^m(M) \cong \mathbb{R}.$$

An explicit isomorphism from $H_c^m(M)$ to the reals is given by

$$H_c^m(M) \to \mathbb{R} : [\omega] \to \int_M \omega.$$

This map is surjective, because M is nonempty, and it is injective by Theorem 7.38.

Lemma 8.31. *For every smooth m-manifold we have*

$$H_c^k(M) \cong H_c^{k+1}(M \times \mathbb{R}), \qquad k = 0, 1, \ldots, m.$$

Corollary 8.32. *The compactly supported deRham cohomology of* $\mathbb{R}^m$ *is given by*

$$H_c^k(\mathbb{R}^m) = \begin{cases} \mathbb{R}, & \text{for } k = m, \\ 0, & \text{for } k < m. \end{cases}$$

Proof. This follows from Example 8.28 by induction. The induction step uses Example 8.29 for $k = 0$ and Lemma 8.31 for $k > 0$. □

Proof of Lemma 8.31. As a warmup we consider the case $M = \mathbb{R}^m$ and use the coordinates $(t, x^1, \ldots, x^m)$ on $\mathbb{R} \times \mathbb{R}^m$. Then a (compactly supported) k-form on $\mathbb{R}^m \times \mathbb{R}$ has the form

$$\omega = \sum_{|I|=k-1} \alpha_I(x,t) dx^I \wedge dt + \sum_{|J|=k} \beta_J(x,t) dx^J,$$

where the α_I and β_J are smooth real valued functions on $\mathbb{R}^m \times \mathbb{R}$ (with compact support). Fixing a real number $t \in \mathbb{R}$ we obtain differential forms

$$\alpha_t := \sum_{|I|=k-1} \alpha_I(x,t) dx^I \in \Omega_c^{k-1}(\mathbb{R}^m),$$
$$\beta_t := \sum_{|J|=k} \beta_J(x,t) dx^J \in \Omega_c^k(\mathbb{R}^m).$$

Going to the general case we see that a compactly supported differential form $\omega \in \Omega_c^k(\mathbb{R} \times M)$ can be written as

$$\omega = \alpha_t \wedge dt + \beta_t, \tag{8.11}$$

where $\mathbb{R} \to \Omega_c^{k-1}(M) : t \mapsto \alpha_t$ and $\mathbb{R} \to \Omega_c^k(M) : t \mapsto \beta_t$ are smooth families of differential forms on M such that the set

$$\mathrm{supp}(\omega) = \overline{\bigcup_{t\in\mathbb{R}} \big(\mathrm{supp}(\alpha_t) \cup \mathrm{supp}(\beta_t)\big) \times \{t\}}$$

is compact. The formula in local coordinates shows that the exterior differential of $\omega \in \Omega_c^k(M \times \mathbb{R})$ is given by

$$d\omega = d^{M\times\mathbb{R}}\omega = \Big(d^M\alpha_t + (-1)^k \partial_t \beta_t\Big) \wedge dt + d^M\beta_t. \tag{8.12}$$

Choose a smooth function $e : \mathbb{R} \to \mathbb{R}$ with compact support such that

$$\int_{-\infty}^{\infty} e(t)\,dt = 1$$

and define the operators

$$\pi_* : \Omega_c^{k+1}(\mathbb{R} \times M) \to \Omega_c^k(M), \qquad e_* : \Omega_c^k(M) \to \Omega_c^{k+1}(\mathbb{R} \times M),$$

by

$$\pi_*\omega := \int_{-\infty}^{\infty} \alpha_t\,dt, \qquad e_*\alpha := e(t)\alpha \wedge dt \tag{8.13}$$

for $\omega = \alpha_t \wedge dt + \beta_t \in \Omega_c^{k+1}(M \times \mathbb{R})$ and $\alpha \in \Omega_c^k(M)$. Then it follows from equation (8.12) that

$$\pi_* \circ d = d^M \circ \pi_*, \qquad e_* \circ d^M = d \circ e_*.$$

Hence π_* and e_* induce homomorphisms on compactly supported deRham cohomology, still denoted by π_* and e_*. We have the identity

$$\pi_* \circ e_* = \mathrm{id}$$

both on $\Omega_c^k(M)$ and on $H_c^k(M)$. We prove that the composition $e_* \circ \pi_*$ is chain homotopy equivalent to the identity, namely, that there is an operator

$$K : \Omega_c^{k+1}(M \times \mathbb{R}) \to \Omega_c^k(M \times \mathbb{R})$$

satisfying the identity

$$\mathrm{id} - e_* \circ \pi_* = (-1)^k \big(d \circ K - K \circ d\big) \tag{8.14}$$

on $\Omega_c^{k+1}(M \times \mathbb{R})$. The operator K is given by

$$K\omega := \widetilde{\alpha}_t \wedge dt + \widetilde{\beta}_t, \quad \widetilde{\alpha}_t := 0, \quad \widetilde{\beta}_t := \int_{-\infty}^t \alpha_s\,ds - \int_{-\infty}^t e(s)\,ds \pi_*\omega \tag{8.15}$$

for $\omega = \alpha_t \wedge dt + \beta_t \in \Omega_c^{k+1}(M \times \mathbb{R})$. Combining (8.12) and (8.15) we find

$$\begin{aligned}
dK\omega &= (-1)^k \left(\alpha_t - e(t)\pi_*\omega\right) \wedge dt + d^M \int_{-\infty}^t \alpha_s\,ds - \int_{-\infty}^t e(s)\,ds\, d^M \pi_*\omega, \\
Kd\omega &= \int_{-\infty}^t \Big(d^M \alpha_s + (-1)^{k+1}\partial_s\beta_s\Big)\,ds - \int_{-\infty}^t e(s)\,ds\,\pi_* d\omega \\
&= (-1)^{k+1}\beta_t + d^M \int_{-\infty}^t \alpha_s\,ds - \int_{-\infty}^t e(s)\,ds\, d^M \pi_*\omega.
\end{aligned}$$

Hence

$$dK\omega - Kd\omega = (-1)^k \big(\alpha_t \wedge\, dt + \beta_t - e(t)\pi_*\omega \wedge dt\big) = (-1)^k \big(\omega - e_*\pi_*\omega\big).$$

This proves (8.14) and the lemma. □

8.3.2 The Mayer–Vietoris Sequence for H_c^*

Let M be a smooth m-manifold and $U, V \subset M$ be two open sets such that $U \cup V = M$. The **Mayer–Vietoris sequence** in this setting has the form

$$0 \longleftarrow \Omega_c^k(M) \stackrel{i_*}{\longleftarrow} \Omega_c^k(U) \oplus \Omega_c^k(V) \stackrel{j_*}{\longleftarrow} \Omega_c^k(U \cap V) \longleftarrow 0, \tag{8.16}$$

where $i_* : \Omega_c^k(U) \oplus \Omega_c^k(V) \to \Omega_c^k(M)$ and $j_* : \Omega_c^k(U \cap V) \to \Omega_c^k(U) \oplus \Omega_c^k(V)$ are defined by

$$i_*(\omega_U, \omega_V) := \omega_U + \omega_V, \qquad j_*\omega := (-\omega, \omega)$$

for $\omega_U \in \Omega_c^k(U)$, $\omega_V \in \Omega_c^k(V)$, and $\omega \in \Omega_c^k(U \cap V)$. Here the first summand in the pair $(-\omega, \omega) \in \Omega_c^k(U) \oplus \Omega_c^k(V)$ is understood in the first component as the extension of $-\omega$ to all of U by setting it zero on $U \setminus V$ and in the second component as the extension of ω to all of V by setting it zero on $V \setminus U$. Likewise, the k-form $\omega_U + \omega_V \in \Omega_c^k(M)$ is understood as the sum after extending ω_U to all of M by setting it zero on $V \setminus U$ and extending ω_V to all of M by setting it zero on $U \setminus V$.

Lemma 8.33. *The Mayer–Vietoris sequence* (8.16) *is exact.*

Proof. That j_* is injective is obvious. That the image of j_* agrees with the kernel of i_* follows from the fact that if the sum of the compactly supported differential form $\omega_U \in \Omega_c^k(U)$ and $\omega_V \in \Omega^k(V)$ vanishes on all of M, then the compact set $\mathrm{supp}(\omega_V) = \mathrm{supp}(\omega_U)$ is contained in $U \cap V$.

We prove that i_* is surjective. As in the proof of Lemma 8.14 we choose a partition of unity subordinate to the cover $M = U \cup V$, consisting of two smooth functions $\rho_U : M \to [0, 1]$ and $\rho_V : M \to [0, 1]$ satisfying

$$\mathrm{supp}(\rho_U) \subset U, \qquad \mathrm{supp}(\rho_V) \subset V, \qquad \rho_U + \rho_V \equiv 1.$$

Let $\omega \in \Omega_c^k(M)$ and define $\omega_U \in \Omega_c^k(U)$ and $\omega_V \in \Omega_c^k(V)$ by

$$\omega_U := \rho_U \omega|_U, \qquad \omega_V := \rho_V \omega|_V.$$

Then

$$i_*(\omega_U, \omega_V) = \omega_U + \omega_V = \omega.$$

This proves the lemma. □

As in Section 8.2 we have that i_* and j_* are cochain maps so that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \Omega_c^{k+1}(M) & \xleftarrow{i_*} & \Omega_c^{k+1}(U)\oplus\Omega_c^{k+1}(V) & \xleftarrow{j_*} & \Omega_c^{k+1}(U\cap V) & \longleftarrow & 0\,. \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \longleftarrow & \Omega_c^{k}(M) & \xleftarrow{i_*} & \Omega_c^{k}(U)\oplus\Omega_c^{k}(V) & \xleftarrow{j_*} & \Omega_c^{k}(U\cap V) & \longleftarrow & 0 \end{array}$$

The boundary operator

$$d_* : H_c^k(M) \to H_c^{k+1}(U\cap V)$$

for the long exact sequence is is defined as follows. Given a closed form $\omega \in \Omega_c^k(M)$ choose a pair $(\omega_U, \omega_V) \in \Omega_c^k(U)\oplus\Omega_c^k(V)$ whose image under i_* is ω; then the pair $(d\omega_U, d\omega_V)$ belongs to the kernel of i_*, because ω is closed, and hence belongs to the image of j_*, by exactness; hence there is a unique $(k+1)$-form $d_*\omega \in \Omega_c^{k+1}(U\cap V)$ whose image under j_* is the given pair $(d\omega_U, d\omega_V)$. As before, this form is closed and its cohomology class in $H_c^{k+1}(U\cap V)$ is independent of the choice of the pair (ω_U, ω_V) used in this construction.

Again, there is an explicit formula for the operator d_* coming from the proof of Lemma 8.33. Namely, we define the linear map

$$d_* : \Omega_c^k(M) \to \Omega_c^{k+1}(U\cap V),$$

by

$$d_*\omega := d\rho_V \wedge \omega|_{U\cap V}. \tag{8.17}$$

This operator is well defined because the 1-form $d\rho_V = -d\rho_U$ is supported in $U\cap V$. Moreover, we have

$$d\circ d_* + d_*\circ d = 0$$

and hence d_* assigns closed forms to closed forms and exact forms to exact forms. Thus d_* descends to a homomorphism on cohomology and the reader may check that this is precisely the homomorphism defined by *diagram chasing* as above.

The homomorphisms on compactly supported deRham cohomology induced by i_*, j_*, d_* give rise to a long exact sequence

$$\cdots H_c^k(M) \xleftarrow{i_*} H_c^k(U)\oplus H_c^k(V) \xleftarrow{j_*} H_c^k(U\cap V) \xleftarrow{d_*} H_c^{k-1}(M)\cdots \tag{8.18}$$

which is also called the **Mayer–Vietoris sequence**.

Theorem 8.34. *The Mayer–Vietoris sequence* (8.18) *is exact.*

Proof. That the composition of any two successive homomorphisms is zero follows directly from the definitions.

We prove that $\ker d_* = \operatorname{im} i_*$. Let $\omega \in \Omega^k_c(M)$ be a closed compactly supported k-form on M such that

$$d_*[\omega] = 0.$$

Then there is a compactly supported k-form $\tau \in \Omega^k_c(U \cap V)$ such that

$$d\tau = d(\rho_V \omega)|_{U\cap V} = -d(\rho_U \omega)|_{U\cap V}.$$

Define $\omega_U \in \Omega^k_c(U)$ and $\omega_V | in \Omega^k_c(V)$ by

$$\omega_U := \begin{cases} \rho_U\omega + \tau & \text{on } U \cap V, \\ \rho_U \omega & \text{on } U \setminus V, \end{cases} \qquad \omega_V := \begin{cases} \rho_V\omega + \tau & \text{on } U \cap V, \\ \rho_V \omega & \text{on } V \setminus U. \end{cases}$$

These forms are closed and have compact support. Moreover, $\omega_U + \omega_V = \omega$ and hence $i_*([\omega_U], [\omega_V]) = [\omega]$. Thus we have proved that $\ker d_* = \operatorname{im} i_*$.

We prove that $\ker i_* = \operatorname{im} j_*$. Let $\omega_U \in \Omega^k_c(U)$ and $\omega_V \in \Omega^k_c(V)$ be compactly supported closed k-forms such that

$$i_*([\omega_U], [\omega_V]) = 0.$$

Then there is a compactly supported $(k-1)$-form $\tau \in \Omega^k_c(M)$ such that

$$d\tau = \begin{cases} \omega_U + \omega_V & \text{on } U \cap V, \\ \omega_U & \text{on } U \setminus V, \\ \omega_V & \text{on } V \setminus U. \end{cases}$$

It follows that the k-form

$$\omega := \omega_V|_{U\cap V} - d(\rho_V\tau)|_{U\cap V} = -\omega_U|_{U\cap V} + d(\rho_U\tau)|_{U\cap V} \in \Omega^k_c(U \cap V)$$

has compact support in $U \cap V$. Moreover, ω is closed and the pair

$$j_*(\omega) = \left(\left\{ \begin{array}{ll} -\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V \end{array} \right\}, \left\{ \begin{array}{ll} \omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U \end{array} \right\} \right) \in \Omega^k_c(U) \oplus \Omega^k_c(V)$$

is cohomologous to (ω_U, ω_V). Hence $j_*([\omega]) = ([\omega_U], [\omega_V])$. Thus we have proved that $\ker i_* = \operatorname{im} j_*$.

We prove that $\ker j_* = \operatorname{im} d_*$. Let $\omega \in \Omega_c^k(U \cap V)$ be a compactly supported closed k-form such that

$$j_*([\omega]) = 0.$$

Then there exist compactly supported $(k-1)$-forms $\tau_U \in \Omega_c^{k-1}(U)$ and $\tau_V \in \Omega_c^{k-1}(V)$ such that

$$d\tau_U := \begin{cases} -\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V, \end{cases} \qquad d\tau_V := \begin{cases} \omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U. \end{cases}$$

Define $\tau \in \Omega_c^{k-1}(M)$ and $\sigma \in \Omega_c^{k-1}(U \cap V)$ by

$$\tau := \begin{cases} \tau_U + \tau_V & \text{on } U \cap V, \\ \tau_U & \text{on } U \setminus V, \\ \tau_V & \text{on } V \setminus U, \end{cases} \qquad \sigma := \rho_V \tau_U - \rho_U \tau_V.$$

Then τ is closed and

$$\rho_V \tau|_{U \cap V} = \tau_V|_{U \cap V} + \sigma,$$

hence

$$d(\rho_V \tau)|_{U \cap V} = d\tau_V|_{U \cap V} + d\sigma = \omega + d\sigma,$$

and hence

$$d_*[\tau] = [d\rho_V \wedge \tau|_{U \cap V}] = [\omega].$$

Thus we have proved that $\ker j_* = \operatorname{im} d_*$. This proves the theorem. □

The proof of Theorem 8.34 also follows from Lemma 8.33 and an abstract general principle in homological algebra, namely, that every short exact sequence of (co)chain complexes determines uniquely a long exact sequence in (co)homology. In the proof of Theorem 8.34 we have established exactness with the boundary map given by an explicit formula. The formulas for the boundary maps d^* and d_* in the Mayer–Vietoris sequences will be useful in the proof of Poincaré duality. The Mayer–Vietoris sequence for compactly supported deRham cohomology can be used as before to establish finite dimensionality and the Künneth formula. This is the content of the next three corollaries.

Corollary 8.35. *If $M = U \cup V$ is the union of two open sets such that the compactly supported deRham cohomology of U, V, $U \cap V$ is finite dimensional, then so is the compactly supported deRham cohomology of M.*

Proof. The proof is exactly the same as that of Corollary 8.16. □

Corollary 8.36. *If M admits a finite good cover then its compactly supported deRham cohomology is finite dimensional.*

Proof. The proof is analogous to that of Corollary 8.20, using Corollary 8.32 instead of Example 8.12. □

Corollary 8.37 (Künneth formula). *If M and N have finite good covers then the map $\Omega_c^k(M) \otimes \Omega_c^\ell(N) \to \Omega_c^{k+\ell}(M \times N) : \omega \otimes \tau \mapsto \pi_M^* \omega \wedge \pi_N^* \tau$ induces an isomorphism*

$$\kappa : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N).$$

Thus

$$\bigoplus_{k=0}^{\ell} H_c^k(M) \otimes H_c^{\ell-k}(N) \cong H_c^\ell(M \times N)$$

for every integer $\ell \geq 0$ and

$$\dim H_c^*(M \times N) = \dim H_c^*(M) \cdot \dim H_c^*(N).$$

Proof. The proof is exactly the same as that of Theorem 8.22. □

8.4 Poincaré Duality

8.4.1 The Poincaré Pairing

Let M be an oriented smooth m-dimensional manifold without boundary. Then, for every integer $k \in \{0, 1, \dots, m\}$, there is a bilinear map

$$\Omega^k(M) \times \Omega_c^{m-k}(M) : (\omega, \tau) \mapsto \int_M \omega \wedge \tau. \tag{8.19}$$

If ω and τ are closed and one of the forms ω and τ is exact (which in the case of τ means that it is the exterior differential of a *compactly supported* $(m-k-1)$-form) then $\omega \wedge \tau$ is the exterior differential of a compactly supported $(m-1)$-form and hence its integral vanishes, by Theorem 7.26. This shows that the pairing (8.19) descends to a bilinear form on deRham cohomology

$$H^k(M) \times H_c^{m-k}(M) : ([\omega], [\tau]) \mapsto \int_M \omega \wedge \tau. \tag{8.20}$$

called the **Poincaré pairing**.

Theorem 8.38 (Poincaré duality). *Let M be an oriented smooth m-dimensional manifold without boundary and suppose that M has a finite good cover. Then the Poincaré pairing* (8.20) *is nondegenerate. This means the following.*

(a) *If $\omega \in \Omega^k(M)$ is closed and*

$$\tau \in \Omega_c^{m-k}(M), \quad d\tau = 0 \qquad \Longrightarrow \qquad \int_M \omega \wedge \tau = 0$$

then ω is exact.

(b) *If $\tau \in \Omega_c^{m-k}(M)$ is closed and*

$$\omega \in \Omega^k(M), \quad d\omega = 0 \qquad \Longrightarrow \qquad \int_M \omega \wedge \tau = 0$$

then there is a $\sigma \in \Omega_c^{m-k-1}(M)$ such that $d\sigma = \tau$.

Remark 8.39. The assumption that ω is closed is not needed in (a) and the assumption that τ is closed is not needed in (b). In fact if $\int_M \omega \wedge d\sigma = 0$ for every $\sigma \in \Omega_c^{m-k-1}(M)$ then, by Stoke's Theorem 7.26, we have $\int_M d\omega \wedge \sigma = 0$ for every $\sigma \in \Omega_c^{m-k-1}(M)$ and hence $d\omega = 0$. Similarly for τ.

Remark 8.40. The Poincaré pairing (8.20) induces a homomorphism

$$\mathrm{PD} : H^k(M) \to H_c^{m-k}(M)^* = \mathrm{Hom}(H_c^{m-k}(M), \mathbb{R}) \tag{8.21}$$

which assigns to the cohomology class of a closed k-form $\omega \in \Omega^k(M)$ the homomorphism

$$H_c^{m-k}(M) \longrightarrow \mathbb{R} : [\tau] \mapsto \mathrm{PD}([\omega])([\tau]) := \int_M \omega \wedge \tau.$$

Condition (a) says that the homomorphism PD is injective and, if $H_c^{m-k}(M)$ is finite dimensional, condition (b) says that PD is surjective. This last assertion is an exercise in linear algebra. By Corollary 8.20 and Corollary 8.36 we know already that, under the assumptions of Theorem 8.38, both the deRham cohomology and the compactly supported deRham cohomology of M are finite dimensional. Thus the assertion of Theorem 8.38 can restated in the form that $\mathrm{PD} : H^k(M) \to H_c^{m-k}(M)^*$ is an isomorphism for every k. We say that a manifold M **satisfies Poincaré duality** if PD is an isomorphism.

Remark 8.41. The Poincaré pairing (8.20) also induces a homomorphism

$$\mathrm{PD}^* : H_c^{m-k}(M) \to H^k(M)^* = \mathrm{Hom}(H^k(M), \mathbb{R}) \tag{8.22}$$

which sends a class $[\tau] \in H_c^{m-k}(M)$ to the homomorphism

$$H^k(M) \longrightarrow \mathbb{R} : [\omega] \mapsto \mathrm{PD}^*([\tau])([\omega]) := \int_M \omega \wedge \tau.$$

If both $H^k(M)$ and $H_c^{m-k}(M)$ are finite dimensional then (8.21) is bijective if and only if (8.22) is bijective. However, in general these two assertions are not equivalent. It turns out that the operator (8.21) is an isomorphism for every oriented manifold M without boundary while (8.22) is not always an isomorphism. (See [2, Remark 5.7].)

Remark 8.42. If M is compact without boundary then $H_c^*(M) = H^*(M)$. In this case the homomorphisms $\mathrm{PD} : H^k(M) \to H^{m-k}(M)^*$ in (8.21) and $\mathrm{PD}^* : H^k(M) \to H^{m-k}(M)^*$ in (8.22) differ by a sign $(-1)^{k(m-k)}$.

Example 8.43. As a warmup we show that Poincaré duality holds for $M = \mathbb{R}^m$. That $\mathrm{PD} : H^k(\mathbb{R}^m) \to H_c^{m-k}(\mathbb{R}^m)^*$ is an isomorphism for $k > 0$ follows from the fact both cohomology groups vanish. (See Example 8.12 and Corollary 8.32.) For $k = 0$ the Poincaré pairing has the form

$$\Omega^0(\mathbb{R}^m) \times \Omega_c^m(\mathbb{R}^m) : (f, \tau) \mapsto \int_{\mathbb{R}^m} f\tau.$$

If $f \in \Omega^0(\mathbb{R}^m)$ and $\int_M f\tau = 0$ for every compactly supported m-form on M then f vanishes; otherwise $f \neq 0$ on some nonempty open set $U \subset \mathbb{R}^m$ and we can choose

$$\tau = \rho f dx^1 \wedge \cdots \wedge dx^m,$$

where $\rho : \mathbb{R}^m \to \mathbb{R}^+$ is a smooth cutoff function with support in U such that $\rho(x) > 0$ for some $x \in U$; then

$$\int_{\mathbb{R}^m} f\tau = \int_{\mathbb{R}^m} f^2(x)\rho(x) dx^1 \cdots dx^m > 0,$$

a contradiction. Conversey, if $\tau \in \Omega_c^m(\mathbb{R}^m)$ is given such that $\int_{\mathbb{R}^m} f\tau = 0$ for every constant function $f : M \to \mathbb{R}$ then

$$\int_{\mathbb{R}^m} \tau = 0$$

and hence it follows from Theorem 7.38 that there is a compactly supported $(m-1)$-form $\sigma \in \Omega_c^{m-1}(\mathbb{R}^m)$ such that $d\sigma = \tau$.

8.4.2 Proof of Poincaré Duality

Proof of Theorem 8.38. The proof is by induction on the number n of elements in a good cover of M. If $n = 1$ then M is diffeomorphic to $\mathbb{R}^m$ and hence the assertion follows from Example 8.43. Now let $n \geq 2$, suppose that

$$M = U_1 \cup \cdots \cup U_n$$

is a good cover, and suppose that Poincaré duality holds for every oriented m-manifold with a good cover by at most $n-1$ open sets. Denote by $U, V \subset M$ the open sets

$$U := U_1 \cup \cdots \cup U_{n-1}, \qquad V := U_n.$$

Then the induction hypothesis asserts that Poincaré duality holds for the manifolds U, V, and $U \cap V$. We shall prove that M satisfies Poincaré duality by considering simultaneously the Mayer–Vietoris sequences for H^* and H_c^* associated to the cover $M = U \cup V$.

Thus we have commuting diagrams

$$\begin{array}{ccccc} H^k(M) & \xrightarrow{i^*} & H^k(U) \oplus H^k(V) & \xrightarrow{j^*} & H^k(U \cap V) \\ \times & & \times & & \times \\ H_c^{m-k}(M) & \xleftarrow{i_*} & H_c^{m-k}(U) \oplus H_c^{m-k}(V) & \xleftarrow{j_*} & H_c^{m-k}(U \cap V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R} & & \mathbb{R} & & \mathbb{R} \end{array} \tag{8.23}$$

and

$$\begin{array}{ccc} H^k(U \cap V) & \xrightarrow{d^*} & H^{k+1}(M) \\ \times & & \times \\ H_c^{m-k}(U \cap V) & \xleftarrow{\pm d_*} & H_c^{m-k-1}(M) \\ \downarrow & & \downarrow \\ \mathbb{R} & & \mathbb{R} \end{array} \tag{8.24}$$

Commutativity of the first square (8.23) asserts that, for all closed forms $\omega \in \Omega^k(M)$ and $\tau_U \in \Omega_c^{m-k}(U)$, $\tau_V \in \Omega_c^{m-k}(V)$ we have

$$\int_M \omega \wedge i_*(\tau_U, \tau_V) = \int_U \omega|_U \wedge \tau_U + \int_V \omega|_V \wedge \tau_V.$$

This follows from the definition of $i_* : \Omega_c^{m-k}(U) \oplus \Omega_c^{m-k}(V) \to \Omega_c^{m-k}(M)$. (See (8.16).) Commutativity of the second square (8.23) asserts that, for all closed forms $\omega_U \in \Omega^k(U)$, $\omega_V \in \Omega^{m-k}(V)$, and $\tau \in \Omega_c^{m-k}(U \cap V)$ we have

$$\int_U \omega_U \wedge (-\tau) + \int_V \omega_V \wedge \tau = \int_{U \cap V} j^*(\omega_U, \omega_V) \wedge \tau.$$

This follows from the definition of $j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$. (See (8.6).) Commutativity of the diagram (8.24) asserts that, for all closed forms $\omega \in \Omega^k(U \cap V)$ and $\tau \in \Omega_c^{m-k-1}(M)$, we have

$$\pm \int_{U \cap V} \omega \wedge d_* \tau = \int_M d^* \omega \wedge \tau.$$

To see this, we recall that

$$d^*\omega = d\rho_V \wedge \omega \in \Omega^{k+1}(M), \qquad d_*\tau = d\rho_U \wedge \tau \in \Omega_c^{m-k}(U \cap V).$$

Here $d\rho_V \wedge \omega$ is extended to all of M by setting it equal to zero on $M \setminus (U \cap V)$, and $d\rho_U \wedge \tau$ is restricted to $U \cap V$ where it still has compact support. Since $d\rho_U + d\rho_V = 0$ we obtain

$$\begin{aligned} \int_M d^*\omega \wedge \tau &= \int_{U \cap V} d\rho_V \wedge \omega \wedge \tau \\ &= (-1)^k \int_{U \cap V} \omega \wedge d\rho_V \wedge \tau \\ &= (-1)^{k+1} \int_{U \cap V} \omega \wedge d\rho_U \wedge \tau \\ &= (-1)^{k+1} \int_{U \cap V} \omega \wedge d_*\tau \end{aligned}$$

as claimed. With the commutativity of (8.23) and (8.24) established, we obtain a commuting diagram

$$\begin{array}{ccccccc} H^k(M) & \longrightarrow & \begin{array}{c} H^k(U) \\ \oplus \\ H^k(V) \end{array} & \longrightarrow & H^k(U \cap V) & \longrightarrow & H^{k+1}(M) \\ \downarrow \text{PD} & & \cong \downarrow \text{PD} & & \cong \downarrow \text{PD} & & \downarrow \text{PD} \\ H_c^{m-k}(M)^* & \longrightarrow & \begin{array}{c} H_c^{m-k}(U)^* \\ \oplus \\ H_c^{m-k}(V)^* \end{array} & \longrightarrow & H_c^{m-k}(U \cap V)^* & \longrightarrow & H_c^{m-k-1}(M) \end{array}$$

Since the horizontal sequences are exact and the Poincaré duality homomorphisms $\text{PD} : H^* \to H_c^{m-*}$ are isomorphisms for U, V, and $U \cap V$, it follows from the Five Lemma 8.23 that $\text{PD} : H^*(M) \to H_c^{m-*}(M)$ is an isomorphism as well. This proves the theorem. □

8.4.3 Poincaré Duality and Intersection Numbers

Let M be a compact oriented smooth m-manifold without boundary so that the compactly supported deRham cohomology of M agrees with the usual deRham cohomology. Then Theorem 8.38 asserts that every linear map $\phi : H^{m-k}(M) \to \mathbb{R}$ determines a unique deRham cohomology class $[\tau] \in H^k(M)$ such that $\phi([\omega]) = \int_M \omega \wedge \tau$ for every closed form $\omega \in \Omega^k(M)$. An important class of examples of such homomorphisms ϕ arises from integration over submanifolds, or more generally, from the integration of pullbacks under smooth maps. More precisely, let P be a compact oriented manifold of dimension $\dim P = m - k$ and let $f : P \to M$ be a smooth map. Then there is a closed form $\tau_f \in \Omega^k(M)$, unique up to an additive exact form, such that

$$\int_M \omega \wedge \tau_f = \int_P f^*\omega \tag{8.25}$$

for every closed form $\omega \in \Omega^{m-k}(M)$. This follows immediately from Theorem 8.38 and Remark 8.41. Namely, the deRham cohomology class of τ_f in $H^k(M)$ is the inverse of the linear map $H^{m-k}(M) \to \mathbb{R} : [\omega] \mapsto \int_P f^*\omega$ under isomorphism $\mathrm{PD}^* : H^k(M) \to H^{m-k}(M)^*$ in (8.22). The unique deRham cohomology class $[\tau_f] \in H^k(M)$ is called **dual to** f. We also call each representative of this class dual to f. If $Q \subset M$ is a compact oriented submanifold without boundary of dimension $\dim Q = m - \ell$ we use this construction for the obvious embedding of Q into M. Thus there is a closed form $\tau_Q \in \Omega^\ell(M)$, unique up to an additive exact form, such that

$$\int_M \omega \wedge \tau_Q = \int_Q \omega \tag{8.26}$$

for every closed form $\omega \in \Omega^{m-\ell}(M)$. The unique deRham cohomology class $[\tau_Q] \in H^\ell(M)$ of such a form as well as the forms τ_Q themselves are called **dual to** Q. The next theorem relates the cup product to intersection theory. The proof will be given in Section 9.2.4.

Theorem 8.44. *Let M and P be compact oriented smooth manifolds without boundary, $f : P \to M$ be a smooth map, and $Q \subset M$ be a compact oriented submanifold without boundary. Assume*

$$\dim P = m - k, \qquad \dim Q = m - \ell, \qquad \dim M = m = k + \ell$$

and let $\tau_f \in \Omega^k(M)$ and $\tau_Q \in \Omega^\ell(M)$ be closed forms dual to f and Q, respectively. Then the intersection number of f and Q is given by

$$f \cdot Q = \int_M \tau_f \wedge \tau_Q = \int_Q \tau_f = (-1)^{k\ell} \int_P f^*\tau_Q. \tag{8.27}$$

8.4.4 Euler Characteristic and Betti Numbers

The **Betti numbers** of a manifold are defined as the dimensions of the deRham cohomology groups and are denoted by

$$b_i := \dim H^i(M), \qquad i = 0, \ldots, m.$$

By Corollary 8.21 these numbers are finite whenever M is compact. Recall that we have defined the **Euler characteristic** $\chi(M)$ of a compact manifold M without boundary as the sum of the indices of a vector field with only isolated zeros. The next theorem shows that this invariant agrees with the alternating sum of the Betti numbers (under the assumption that M is oriented). It shows also that the Lefschetz number of a smooth map form M to itself (defined as the sum of the fixed point indices) can be expressed in terms of the traces of the induced maps on deRham cohomology.

Theorem 8.45. *Let M be a compact oriented smooth manifold without boundary and let $f : M \to M$ be a smooth map. Then the Euler characteristic of M is given by*

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \dim H^i(M) \tag{8.28}$$

and the Lefschetz number of f is given by

$$L(f) = \sum_{i=0}^{m} (-1)^i \mathrm{trace}\left(f^* : H^i(M) \to H^i(M)\right). \tag{8.29}$$

Proof. Choose closed differential forms

$$\omega_i \in \Omega^{k_i}(M), \qquad d\omega_i = 0, \qquad i = 0, 1, \ldots, n,$$

whose cohomology classes $[\omega_i]$ form a basis of $H^*(M)$. By Theorem 8.38 there is a dual basis

$$\tau_j \in \Omega^{m-k_j}(M), \qquad d\tau_j = 0, \qquad j = 0, 1, \ldots, n,$$

such that

$$\int_M \omega_i \wedge \tau_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

(Let $\eta_j : H^{k_j}(M) \to \mathbb{R}$ be the linear functional that sends $[\omega_i]$ to δ_{ij} for each i with $k_i = k_j$ and choose a closed form $\tau_j \in \Omega^{m-k_j}$ such that $\mathrm{PD}^*([\tau_j]) = \eta_j$.)

Now consider the manifold $M \times M$ and denote by $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ the projections onto the first and second factors. By Theorem 8.22 the cohomology classes of the forms $\pi_1^*\omega_i \wedge \pi_2^*\tau_j$ form a basis of the deRham cohomology of $M \times M$. In particular, there are real numbers $c_{ij} \in \mathbb{R}$ such that the cohomology class $[\tau_\Delta] \in H^m(M \times M)$, dual to the diagonal $\Delta \subset M \times M$ as in Section 8.4.3, can be expressed in the form

$$[\tau_\Delta] = \sum_{i,j} c_{ij}[\pi_1^*\omega_i \wedge \pi_2^*\tau_j]. \tag{8.30}$$

We compute the coefficients c_{ij} by using the equation

$$\int_\Delta \omega = \int_{M\times M} \omega \wedge \tau_\Delta, \qquad \omega := \pi_1^*\tau_\ell \wedge \pi_2^*\omega_k,$$

If $\iota : M \to M\times M$ denotes the inclusion of the diagonal given by $\iota(p) := (p,p)$ for $p \in M$ then $\pi_1 \circ \iota = \pi_2 \circ \iota = \mathrm{id}$ and hence

$$\int_\Delta \omega = \int_M \iota^*(\pi_1^*\tau_k \wedge \pi_2^*\tau_\ell) = \int_M \tau_\ell \wedge \omega_k = (-1)^{\deg(\omega_k)\deg(\tau_\ell)}\delta_{k\ell}.$$

Moreover, by (8.30), we have

$$\begin{aligned}
\int_{M\times M} \omega \wedge \tau_\Delta &= \sum_{i,j} c_{ij} \int_{M\times M} \pi_1^*\tau_\ell \wedge \pi_2^*\omega_k \wedge \pi_1^*\omega_i \wedge \pi_2^*\tau_j \\
&= \sum_{i,j} c_{ij}(-1)^{\deg(\omega_k)\deg(\omega_i)} \int_{M\times M} \pi_1^*\tau_\ell \wedge \pi_1^*\omega_i \wedge \pi_2^*\omega_k \wedge \pi_2^*\tau_j \\
&= \sum_{i,j} c_{ij}(-1)^{\deg(\omega_k)\deg(\omega_i)} \int_M \tau_\ell \wedge \omega_i \int_M \omega_k \wedge \tau_j \\
&= \sum_{i,j} c_{ij}(-1)^{\deg(\omega_k)\deg(\omega_i)}(-1)^{\deg(\omega_k)\deg(\tau_j)}\delta_{i\ell}\delta_{jk} \\
&= (-1)^{\deg(\omega_k)\deg(\omega_\ell)}(-1)^{\deg(\omega_k)\deg(\tau_k)}c_{\ell k}
\end{aligned}$$

Setting $k = \ell$ we find

$$c_{k\ell} = (-1)^{\deg(\omega_k)}\delta_{k\ell}$$

and hence, by (8.30),

$$[\tau_\Delta] = \sum_i (-1)^{\deg(\omega_i)}[\pi_1^*\omega_i \wedge \pi_2^*\tau_i]. \tag{8.31}$$

Now the Lefschetz number of f is given by

$$L(f) = \mathrm{graph}(f) \cdot \Delta.$$

We can express this number in terms of Theorem 8.44 with M replaced by the product manifold $M \times M$, with

$$Q = \Delta \subset M \times M,$$

and with $f : P \to M$ replaced by the map

$$\mathrm{id} \times f : M \to M \times M.$$

Since

$$\pi_1 \circ (\mathrm{id} \times f) = \mathrm{id} : M \to M, \qquad \pi_2 \circ (\mathrm{id} \times f) = f : M \to M$$

we obtain

$$\begin{aligned}
L(f) &= \mathrm{graph}(f) \cdot \Delta \\
&= (-1)^m \int_M (\mathrm{id} \times f)^* \tau_\Delta \\
&= (-1)^m \sum_i (-1)^{\deg(\omega_i)} \int_M (\mathrm{id} \times f)^* (\pi_1^* \omega_i \wedge \pi_2^* \tau_i) \\
&= (-1)^m \sum_i (-1)^{\deg(\omega_i)} \int_M \omega_i \wedge f^* \tau_i \\
&= \sum_{k=0}^m (-1)^k \sum_{\deg(\tau_i)=k} \int_M \omega_i \wedge f^* \tau_i \\
&= \sum_{k=0}^m (-1)^k \mathrm{trace}\big(f^* : H^k(M) \to H^k(M)\big).
\end{aligned}$$

Here the last equation follows from the fact that

$$f^* \tau_i = \sum_{\deg(\tau_j)=k} a_{ij} \tau_j, \qquad a_{ij} := \int_M \omega_j \wedge f^* \tau_i,$$

whenever $\deg(\tau_i) = k$, and hence

$$\mathrm{trace}\big(f^* : H^k(M) \to H^k(M)\big) = \sum_{\deg(\tau_i)=k} a_{ii} = \sum_{\deg(\tau_i)=k} \int_M \omega_i \wedge f^* \tau_i.$$

This proves (8.29). The Euler characteristic of M is equal to the Lefschetz number of the identity map on M and hence (8.28) follows immediately from (8.29). This proves the theorem. □

Remark 8.46. The **zeta function** of a smooth map

$$f : M \to M$$

on a compact oriented m-manifold M without boundary (thought of as a discrete-time dynamical system) is defined by

$$\zeta_f(t) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)t^n}{n}\right), \tag{8.32}$$

where

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}} : M \to M$$

denotes the nth iterate of f. By definition of the Lefschetz numbers (in terms of an algebraic count of the fixed points) the zeta-function of f can be expressed in terms a count of the periodic points of f, provided that they are all isolated. If the periodic points of f are all nondegenerate then the zeta-function of f can be written in the form

$$\zeta_f(t) = \prod_{n=1}^{\infty} \prod_{p \in \mathcal{P}_n(f)/\mathbb{Z}_n} (1 - \varepsilon(p, f^n)t^n)^{-\varepsilon(p,f^n)\iota(p,f^n)}, \tag{8.33}$$

where $\mathcal{P}_n(f)$ denotes the set of periodic points with minimal period n and

$$\iota(p, f^n) := \operatorname{sign}\det(\mathbb{1} - df^n(p)),$$
$$\varepsilon(p, f^n) := \operatorname{sign}\det(\mathbb{1} + df^n(p))$$

for $p \in \mathcal{P}_n(f)$. This formula is due to Ionel and Parker. One can use Theorem 8.45 to prove that

$$\begin{aligned}\zeta_f(t) &= \prod_{i=0}^{m} \det\left(\mathbb{1} - tf^* : H^i(M) \to H^i(M)\right)^{(-1)^{i+1}} \\ &= \frac{\det\left(\mathbb{1} - tf^* : H^{\mathrm{odd}}(M) \to H^{\mathrm{odd}}(M)\right)}{\det\left(\mathbb{1} - tf^* : H^{\mathrm{ev}}(M) \to H^{\mathrm{ev}}(M)\right)}.\end{aligned} \tag{8.34}$$

In particular, the zeta function is rational.

Exercise 8.47. Prove that the right hand side of (8.32) converges for t sufficiently small. Prove the formulas (8.33) and (8.34). **Hint:** Use the identities

$$\det(\mathbb{1} - tA)^{-1} = \exp\left(\operatorname{trace}\left(\sum_{n=1}^{\infty} \frac{t^n A^n}{n}\right)\right), \qquad \iota(p, f^n) = \iota(p, f)\varepsilon(p, f)^{n-1}$$

for a square matrix A and $t \in \mathbb{R}$ sufficiently small, and for a fixed point p of f that is nondegenerate for all iterates of f.

8.4.5 Examples and Exercises

Example 8.48 (The DeRham Cohomology of the Torus). It follows from the Künneth formula in Theorem 8.22 by induction that the deRham cohomology of the m-torus

$$\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m \cong \underbrace{S^1 \times \cdots \times S^1}_{m \text{ times}}$$

has dimension

$$\dim H^k(\mathbb{T}^m) = \binom{m}{k}.$$

Hence every k-dimensional deRham cohomology class can be represented uniquely by a k-form

$$\omega_c = \sum_{1 \le i_1 < \cdots < i_k \le m} c_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

with constant coefficients. Thus the map $c \mapsto [\omega_c]$ defines an isomorphism

$$\Lambda^*(\mathbb{R}^m)^* \to H^*(\mathbb{T}^m).$$

This is an isomorphism of algebras with the exterior product on the left and the cup product on the right.

Exercise 8.49. Show that a closed k-form $\omega \in \Omega^k(\mathbb{T}^m)$ is exact if and only if its integral vanishes over every compact oriented k-dimensional submanifold of $\mathbb{T}^m$. **Hint:** Given a closed k-form $\omega \in \Omega^k(\mathbb{T}^m)$ choose c such that $\omega - \omega_c$ is exact. Express the number $c_{i_1 \cdots i_k}$ as an integral of ω over a k-dimensional subtorus of $\mathbb{T}^m$.

Exercise 8.50. Prove that a 1-form $\omega \in \Omega^1(M)$ is exact if and only if its integral vanishes over every smooth loop in M. Show that every connected simply connected manifold M satisfies

$$H^1(M) = 0.$$

Hint: Assume $\omega \in \Omega^1(M)$ satisfies $\int_{S^1} \gamma^*\omega = 0$ for every smooth map $\gamma : S^1 \to M$. Fix a point $p_0 \in M$ and define the function $f : M \to \mathbb{R}$ as follows. Given $p \in M$ choose a smooth path $\gamma : [0,1] \to M$ such that $\gamma(0) = p_0$ and $\gamma(1) = p$ and define

$$f(p) := \int_{[0,1]} \gamma^*\omega.$$

Prove that the value $f(p)$ does not depend on the choice of the path γ. Prove that f is smooth. Prove that $df = \omega$.

Example 8.51 (The Genus of a Surface). Let Σ be a compact connected oriented 2-manifold without boundary. Then Theorem 8.38 asserts that the Poincaré pairing

$$H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R} : ([\alpha], [\beta]) \mapsto \int_\Sigma \alpha \wedge \beta$$

is nondegenerate. Since this pairing is skew-symetric it follows that $H^1(\Sigma)$ is even dimensional. Hence there is a nonnegative integer $g \in \mathbb{N}_0$, called the **genus of** Σ, such that

$$\dim H^1(\Sigma) = 2g.$$

Moreover, since Σ is connected, we have $H^0(\Sigma) = \mathbb{R}$ and $H^2(\Sigma) = \mathbb{R}$ (see Theorem 7.38 or Theorem 8.38). Hence, by Theorem 8.45, the Euler characteristic of Σ is given by

$$\chi(\Sigma) = 2 - 2g.$$

Thus the Euler characteristic is even and less than or equal to two. Since the 2-sphere is simply connected we have $H^1(S^2) = 0$, by Exercise 8.50, and hence the 2-sphere has genus zero and Euler characteristic two. This follows also from the Poincaré–Hopf Theorem. By Example 8.48 the 2-torus has genus one and Euler characteristic zero. This can again be derived from the Poincaré–Hopf theorem because there is a vector field on the torus without zeros. All higher genus surfaces have negative Euler characteristic. Examples of surfaces of genus zero, one, and two are depicted in Figure 8.1. By the Gauss–Bonnet Formula only genus one surfaces can admit flat metrics. A fundamental result in two dimensional differential topology is that two compact connected oriented 2-manifolds without boundary are diffeomorphic if and only if they have the same genus. A beautiful proof of this theorem, based on Morse theory, is contained in the book of Hirsch [6].

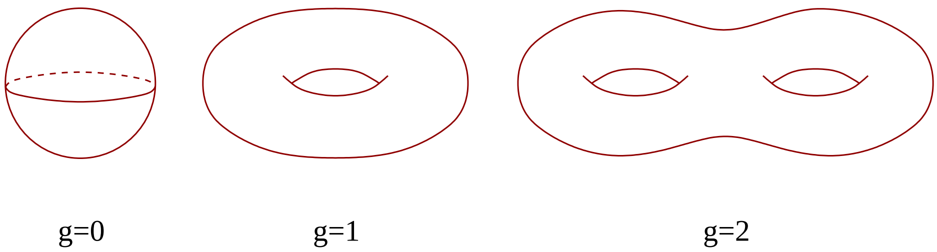


Figure 8.1: The Genus of a Surface.

Example 8.52 (The DeRham Cohomology of $\mathbb{C}\mathrm{P}^n$). The deRham cohomology of $\mathbb{C}\mathrm{P}^n$ is given by

$$H^k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{R}, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}. \end{cases} \tag{8.35}$$

We explain the cup product structure on $H^*(\mathbb{C}\mathrm{P}^n)$ at the end of Chapter 9.

For $\mathbb{C}\mathrm{P}^1 \cong S^2$ the formula (8.35) follows from Example 8.51. We prove the general formula by induction on n. Take $n \geq 2$ and suppose the assertion has been proved for $\mathbb{C}\mathrm{P}^{n-1}$. Consider the open subsets

$$\begin{aligned} U &:= \mathbb{C}\mathrm{P}^n \setminus \{[0 : \cdots : 0 : 1]\}, \\ V &:= \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^{n-1} = \{[z_0 : \cdots : z_{n-1} : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, z_n \neq 0\}. \end{aligned}$$

They cover $\mathbb{C}\mathrm{P}^n$, the set V is diffeomorphic to $\mathbb{C}^n$ and the obvious inclusion $\iota : \mathbb{C}\mathrm{P}^{n-1} \to U$ is a homotopy equivalence. A homotopy inverse of the inclusion is the projection $\pi : U \to \mathbb{C}\mathrm{P}^{n-1}$ given by

$$\pi([z_0 : \cdots : z_{n-1} : z_n]) := [z_0 : \cdots : z_{n-1}]$$

Then $\pi \circ \iota = \mathrm{id} : \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ and $\iota \circ \pi : U \to U$ is homotopic to the identity by the homotopy $f_t : U \to U$ given by

$$f_t([z_0 : \cdots : z_{n-1} : z_n) := [z_0 : \cdots : z_{n-1} : tz_n]$$

with

$$f_0 = \iota \circ \pi, \qquad f_1 = \mathrm{id}.$$

Hence the inclusion $\iota : \mathbb{C}\mathrm{P}^{n-1} \to U$ induces an isomorphism on cohomology, by Corollary 8.5, and the cohomology of V is isomorphic to that of $\mathbb{C}^n$. Thus it follows from the induction hypothesis and Example 8.12 that

$$H^k(U) = \begin{cases} \mathbb{R}, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}, \end{cases} \qquad H^k(V) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{if } k > 0. \end{cases}$$

Moreover, the intersection $U \cap V$ is diffeomorphic to $\mathbb{C}^n \setminus \{0\}$ and therefore is homotopy equivalent to S^{2n-1}. Thus, by Example 8.13, we have

$$H^k(V) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{if } 1 \leq k \leq 2n-2, \\ \mathbb{R}, & \text{if } k = 2n-1. \end{cases}$$

Hence, for $2 \le k \le 2n-2$, the Mayer–Vietoris sequence takes the form

$$\begin{array}{ccccccc} H^{k-1}(U\cap V) & \stackrel{d^*}{\to} & H^k(\mathbb{C}\mathrm{P}^n) & \stackrel{i^*}{\to} & H^k(U)\oplus H^k(V) & \stackrel{j^*}{\to} & H^k(U\cap V) \\ \| & & \| & & \| & & \| \\ 0 & \to & H^k(\mathbb{C}\mathrm{P}^n) & \to & H^k(\mathbb{C}\mathrm{P}^{n-1}) & \to & 0 \end{array}.$$

This sequence is exact, by Theorem 8.15. Hence the inclusion induced homomorphism

$$\iota^* : H^k(\mathbb{C}\mathrm{P}^n) \to H^k(\mathbb{C}\mathrm{P}^{n-1}) \tag{8.36}$$

is an isomorphism for $2 \le k \le 2n-2$. Since $\mathbb{C}\mathrm{P}^n$ is connected, we have

$$H^0(\mathbb{C}\mathrm{P}^n) = \mathbb{R}.$$

Since $\mathbb{C}\mathrm{P}^n$ is simply connected, by Exercise 8.53 below, it follows from Exercise 8.50 that

$$H^1(\mathbb{C}\mathrm{P}^n) = 0.$$

(Exercise: Deduce this instead from the Mayer–Vietoris sequence.) Since $\mathbb{C}\mathrm{P}^n$ is a complex manifold, it is oriented and therefore satisfies Poincaré duality. Hence, by Theorem 8.38 we have

$$H^{2n}(\mathbb{C}\mathrm{P}^n) \cong H^0(\mathbb{C}\mathrm{P}^n) = \mathbb{R}, \qquad H^{2n-1}(\mathbb{C}\mathrm{P}^n) \cong H^1(\mathbb{C}\mathrm{P}^n) = 0.$$

This proves (8.35) for all n. It also follows that the homomorphism (8.36) is an isomorphism for $0 \le k \le 2n-2$.

Exercise 8.53. Prove that $\mathbb{C}\mathrm{P}^n$ is simply connected.

Exercise 8.54 (The DeRham Cohomology of $\mathbb{R}\mathrm{P}^m$). Prove that the deRham cohomology of $\mathbb{R}\mathrm{P}^m$ is

$$H^k(\mathbb{R}\mathrm{P}^m) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{if } 1 \le k \le m-1, \\ 0, & \text{if } k = m \text{ is even}, \\ \mathbb{R}, & \text{if } k = m \text{ is odd}. \end{cases}$$

In particular, $\mathbb{R}\mathrm{P}^2$ has Euler characteristic one. **Hint:** $\mathbb{R}\mathrm{P}^m$ is oriented if and only if m is odd. Prove that, up to homotopy, there is only one noncontractible loop in $\mathbb{R}\mathrm{P}^m$, and hence its fundamental group is isomorphic to $\mathbb{Z}_2$. Use Exercise 8.50 to prove that $H^1(\mathbb{R}\mathrm{P}^m) = 0$ for $m \ge 2$. Use an induction argument and Mayer–Vietoris to prove that $H^k(\mathbb{R}\mathrm{P}^m) = 0$ for $2 \le k \le m-1$.

8.5 The Čech-deRham Complex

In Section 8.2 on the Mayer–Vietoris sequence we have studied the deRham cohmology of a smooth manifold M by restricting global differential forms on M to two open sets and differential forms on the two open sets to their intersection and examining the resulting combinatorics. We have seen that this technique is a powerful tool for understanding deRham cohomology allowing us, for example, to prove finite dimensionality, derive the Künneth formula, and establish Poincaré duality for compact manifolds in an elegant manner. The Mayer–Vietoris principle can be carried over to covers of M by an arbitrarly many (or even infinitely many) open sets. Associated to any open cover (of any topological space) is the Čech cohomology. In general, this cohomology will depend on the choice of the cover. We shall prove that the Čech cohomology of a good cover of a smooth manifold is isomorphic to the deRham cohomology and hence is independent of the choice of the good cover. This result is a key ingredient in the proof of deRham's theorem which asserts that the deRham cohomology of a manifold its isomorphic to the singular cohomology with real coefficients.

8.5.1 The Čech Complex

Let M be a smooth manifold and

$$\mathscr{U} = \{U_i\}_{i\in I}$$

be an open cover of of M, indexed by a set I, such that

$$U_i \neq \emptyset$$

for every $i \in I$. The combinatorics of the cover $\mathscr{U}$ is encoded in the sets of multi-indices associated to nonempty intersections, denoted by

$$\mathcal{I}_k(\mathscr{U}) := \Big\{(i_0,\dots,i_k) \in I^k \,|\, U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset\Big\}$$

for every nonnegative integer k. The permutation group S_{k+1} of bijections of the set $\{0,1,\dots,k\}$ acts on the set $\mathcal{I}_k(\mathscr{U})$ and the nonempty intersections of $k+1$ sets in $\mathscr{U}$ correspond to orbits under this action: reordering the indices doesn't change the intersection. We shall consider ordered nonempty intersections up to even permutations; the convention is that odd permutations act by a sign change on the data associated to an ordered nonempty intersection.

The simplest way of assigning a cochain complex to these data is to assign a real number to each ordered nonempty intersection of $k+1$ sets in $\mathscr{U}$. Thus we assign a real number $c_{i_0\cdots i_k}$ to each ordered $(k+1)$-tuple $(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})$ with the convention that the sign changes under every odd reordering of the indices. In particular, the number $c_{i_0\cdots i_k}$ is zero whenever there is any repetition among the indices and is undefined whenever $U_{i_0}\cap\cdots\cap U_{i_k}=\emptyset$. Let $C^k(\mathscr{U},\mathbb{R})$ denote the real vector space of all tuples

$$c=(c_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}(\mathscr{U})}\in\mathbb{R}^{\mathcal{I}_k(\mathscr{U})}$$

that satisfy the condition

$$c_{i_{\sigma(0)}\cdots i_{\sigma(k)}}=\varepsilon(\sigma)c_{i_0\cdots i_k}$$

for $\sigma\in S_{k+1}$ and $(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})$. These spaces determine a cochain complex

$$C^0(\mathscr{U},\mathbb{R})\stackrel{\delta}{\longrightarrow}C^1(\mathscr{U},\mathbb{R})\stackrel{\delta}{\longrightarrow}C^2(\mathscr{U},\mathbb{R})\stackrel{\delta}{\longrightarrow}C^3(\mathscr{U},\mathbb{R})\stackrel{\delta}{\longrightarrow}\cdots. \tag{8.37}$$

called the **Čech complex of the open cover $\mathscr{U}$ with real coefficients**. The boundary operator $\delta:C^k(\mathscr{U},\mathbb{R})\to C^{k+1}(\mathscr{U},\mathbb{R})$ is defined by

$$(\delta c)_{i_0\cdots i_{k+1}}:=\sum_{\nu=0}^{k+1}(-1)^\nu c_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}} \tag{8.38}$$

for $c=(c_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}(\mathscr{U})}\in C^k(\mathscr{U},\mathbb{R})$.

Example 8.55. A Čech 0-cochain $c\in C^0(\mathscr{U},\mathbb{R})$ assign a real number c_i to every open set U_i, a Čech 1-cochain $c\in C^1(\mathscr{U},\mathbb{R})$ assigns a real number c_{ij} to every nonempty ordered intersection $U_i\cap U_j$ such that

$$c_{ij}=-c_{ji}$$

and a Čech 2-cochain $c\in C^2(\mathscr{U},\mathbb{R})$ assigns a real number c_{ijk} to every nonempty ordered triple intersection $U_i\cap U_j\cap U_k$ such that

$$c_{ijk}=-c_{jik}=-c_{ikj}.$$

The boundary operator δ assigns to a 0-cochain $c=(c_i)_{i\in I}$ the 1-cochain

$$(\delta c)_{ij}=c_j-c_i,\qquad U_i\cap U_j\neq\emptyset,$$

and it assigns to every 1-cochain $c=(c_{ij})_{(i,j)\in\mathcal{I}_1(\mathscr{U})}$ the 2-cochain

$$(\delta c)_{ijk}=c_{jk}+c_{ki}+c_{ij},\qquad U_i\cap U_j\cap U_k\neq\emptyset.$$

One verifies immediately that $\delta\circ\delta=0$. This continues to hold in general as the next lemma shows.

Lemma 8.56. *The image of the linear map $\delta : C^k(\mathscr{U},\mathbb{R}) \to \mathbb{R}^{\mathcal{I}_{k+1}(\mathscr{U})}$ is contained in the subspace $C^{k+1}(\mathscr{U},\mathbb{R})$ and $\delta\circ\delta = 0$.*

Proof. The first assertion is left as an exercise for the reader. To prove the second assertion we choose a k-cochain $c \in C^k(\mathscr{U},\mathbb{R})$ and a multi-index $(i_0,\dots,i_{k+2}) \in \mathcal{I}_{k+2}(\mathscr{U})$ and compute

$$\begin{aligned}
\delta(\delta c)_{i_0\cdots i_{k+2}} &= \sum_{\nu=0}^{k+2}(-1)^\nu(\delta c)_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}} \\
&= \sum_{0\le\mu<\nu\le k+2}(-1)^{\nu+\mu}c_{i_0\cdots\widehat{i_\mu}\cdots\cdots\widehat{i_\nu}\cdots i_{k+1}} \\
&\quad + \sum_{0\le\nu<\mu\le k+2}(-1)^{\nu+\mu-1}c_{i_0\cdots\widehat{i_\nu}\cdots\cdots\widehat{i_\mu}\cdots i_{k+1}} \\
&= 0.
\end{aligned}$$

This proves the lemma. □

The cohomology of the Čech complex (8.37) is called the **Čech cohomology of $\mathscr{U}$ with real coefficients** and will be denoted by

$$H^k(\mathscr{U},\mathbb{R}) := \frac{\ker\delta : C^k(\mathscr{U},\mathbb{R}) \to C^{k+1}(\mathscr{U},\mathbb{R})}{\operatorname{im}\delta : C^{k-1}(\mathscr{U},\mathbb{R}) \to C^k(\mathscr{U},\mathbb{R})}. \tag{8.39}$$

This beautiful and elementary combinatorial construction works for every open cover of every topological space M and immediately gives rise to the following fundamental questions.

Question 1: *To what extent does the Čech cohomology $H^*(\mathscr{U},\mathbb{R})$ depend on the choice of the open cover?*

Question 2: *If M is a manifold, what is the relation between $H^*(\mathscr{U},\mathbb{R})$ and the deRham cohomology $H^*(M)$ (or any other (co)homology theory)?*

Example 8.57. The Čech cohomology group $H^0(\mathscr{U},\mathbb{R})$ is the kernel of the operator $\delta : C^0(\mathscr{U},\mathbb{R}) \to C^1(\mathscr{U},\mathbb{R})$ and hence consists of all tuples $c = (c_i)_{i\in I}$ that satisfy $c_i = c_j$ whenever $U_i \cap U_j \neq \emptyset$. This shows that, for every Čech 0-cocycle $c = (c_i)_{i\in I} \in H^0(\mathscr{U},\mathbb{R})$, there is a locally constant function $f : M \to \mathbb{R}$ such that $f|_{U_i} \equiv c_i$ for every $i \in I$. If each open set U_i is connected $H^0(\mathscr{U},\mathbb{R})$ is the vector space of all locally constant real valued functions on M and hence

$$H^0(\mathscr{U},\mathbb{R}) = \mathbb{R}^{\pi_0(M)} = H^0(M),$$

where $\pi_0(M)$ is the set of components of M and $H^0(M)$ is the deRham cohomology group. On the other hand, if $\mathscr{U}$ consists only of one open set $U = M$, then $H^0(\mathscr{U},\mathbb{R}) = \mathbb{R}$ is the set of globally constant functions on M.

8.5.2 The Isomorphism

Let M be a smooth manifold and $\mathscr{U} = \{U_o\}_{i\in I}$ be an open cover of M. We show that there is a natural homomorphism from the Čech cohomology of $\mathscr{U}$ to the deRham cohomology of M. The definition of the homomorphism on the cochain level depends on the choice of a partition of unity $\rho_i : M \to [0,1]$ subordinate to the cover $\mathscr{U} = \{U_i\}_{i\in I}$. Define the linear map

$$C^k(\mathscr{U},\mathbb{R}) \to \Omega^k(M) : c \mapsto \omega_c \tag{8.40}$$

by

$$\omega_c := \sum_{(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})} c_{i_0\cdots i_k}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_k}. \tag{8.41}$$

for $c \in C^k(\mathscr{U},\mathbb{R})$.

Lemma 8.58. *The map* (8.40) *is a chain homomorphism and hence induces a homomorphism on cohomology*

$$H^*(\mathscr{U},\mathbb{R}) \to H^*(M) : [c] \mapsto [\omega_c]. \tag{8.42}$$

Proof. It will sometimes be convenient to set $c_{i_0\cdots i_k} := 0$ for $c \in C^k(\mathscr{U},\mathbb{R})$ and $(i_0,\dots,i_k) \in I^{k+1} \setminus \mathcal{I}_k(\mathscr{U})$. We prove that the map (8.40) is a chain homomorphism. For $c \in C^k(\mathscr{U},\mathbb{R})$ we compute

$$\begin{aligned}
\omega_{\delta c} &= \sum_{(i_0,\dots,i_{k+1})\in\mathcal{I}_{k+1}(\mathscr{U})} (\delta c)_{i_0\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= \sum_{(i_0,\dots,i_{k+1})\in\mathcal{I}_{k+1}(\mathscr{U})} \sum_{\nu=0}^{k+1}(-1)^\nu c_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= \sum_{(i_0,\dots,i_{k+1})\in I^{k+2}} c_{i_1\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&\quad + \sum_{\nu=1}^{k+1}(-1)^\nu \sum_{(i_0,\dots,i_{k+1})\in I^{k+2}} c_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= \sum_{(i_1,\dots,i_{k+1})\in I^{k+1}} c_{i_1\cdots i_{k+1}} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= d\omega_c.
\end{aligned}$$

Here we have used the fact that the respective summand vanishes whenever $(i_0,\dots,i_{k+1}) \notin \mathcal{I}_{k+1}(\mathscr{U})$ and that $\sum_{i\in I} d\rho_i = 0$ and $\sum_{i\in I}\rho_i = 1$. Thus (8.40) is a chain map and this proves the lemma. □

Remark 8.59. Let $c \in C^k(\mathscr{U}, \mathbb{R})$ such that $\delta c = 0$. Then, for all tuples $(i, j, i_1, \ldots, i_k) \in \mathcal{I}_{k+1}(\mathscr{U})$, we have

$$c_{ii_1\cdots i_k} = c_{ji_1\cdots i_k} - \sum_{\nu=1}^{k} (-1)^\nu c_{iji_1\cdots\widehat{i_\nu}\cdots i_k}$$

Multiply by $\rho_j d\rho_{i_1} \wedge \cdots d\rho_{i_k}$ and restrict to U_i. Since $\rho_j d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_k}$ vanishes on U_i whenever $(i, j, i_1, \ldots, i_k) \notin \mathcal{I}_{k+1}(\mathscr{U})$, the resulting equation continues to hold for all tuples $(i, j, i_1, \ldots, i_k) \in I^{k+2}$. Fixing i and taking the sum over all tuples $(j, i_1, \ldots, i_k) \in I^{k+1}$ we find

$$\delta c = 0 \qquad \Longrightarrow \qquad \omega_c|_{U_i} = \sum_{(i_1,\ldots,i_k)\in I^k} c_{ii_1\cdots i_k} d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_k}. \tag{8.43}$$

This gives another proof that ω_c is closed whenever $\delta c = 0$.

The next theorem is the main result of this section. It answers the above questions under suitable assumptions on the cover $\mathscr{U}$.

Theorem 8.60. *If $\mathscr{U}$ is a good cover of M then* (8.42) *is an isomorphism from the Čech cohomology of $\mathscr{U}$ to the deRham cohomology of M*

The proof of Theorem 8.60 will in fact show that, under the assumption that $\mathscr{U}$ is a good cover, the homomorphism (8.42) on cohomology is independent of the choice of the partition of unity used to define it. Moreover, we have the following immediate corollary.

Corollary 8.61. *The Čech cohomology groups with real coefficients associated to two good covers of a smooth manifold are isomorphic.*

If $\mathscr{U}$ is a finite good cover the Čech complex $C^*(\mathscr{U}, \mathbb{R})$ is finite dimensional and hence, so is its cohomology $H^*(\mathscr{U}, \mathbb{R})$. Combining this observation with Theorem 8.60 we obtain another proof that the deRham cohomology is finite dimensional as well.

Corollary 8.62. *If a smooth manifold admits a finite good cover then its deRham cohomology is finite dimensional.*

Following Bott and Tu [2] we explain a proof of Theorem 8.60 that is based on a Mayer–Vietoris argument and involves differential forms of all degrees on the open sets in the cover and their intersections. Thus we build a cochain complex that contains both the deRham complex and the Čech complex as subcomplexes.

8.5.3 The Čech–deRham Complex

Associated to the open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of our m-manifold M is a cochain complex defined as follows. Given two nonnegative integers k and p we introduce the vector space

$$C^k(\mathscr{U}, \Omega^p)$$

of all tuples

$$\omega = (\omega_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})}, \qquad \omega_{i_0\cdots i_k} \in \Omega^p(U_{i_0}\cap\cdots\cap U_{i_k}),$$

that satisfy $\omega_{i_{\sigma(0)}\cdots i_{\sigma(k)}} = \varepsilon(\sigma)\omega_{i_0\cdots i_k}$ for $\sigma \in S_{k+1}$ and $(i_0,\dots,i_k) \in \mathcal{I}_k(\mathscr{U})$. This complex carries two boundary operators

$$\delta : C^k(\mathscr{U}, \Omega^p) \to C^{k+1}(\mathscr{U}, \Omega^p), \qquad d : C^k(\mathscr{U}, \Omega^p) \to C^k(\mathscr{U}, \Omega^{p+1})$$

defined by

$$(\delta\omega)_{i_0\cdots i_{k+1}} := \sum_{\nu=0}^{k+1}(-1)^\nu \omega_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}, \qquad (d\omega)_{i_0\cdots i_{k+1}} := d\omega_{i_0\cdots i_{k+1}}. \quad (8.44)$$

They satisfy the equations

$$\delta\circ\delta = 0, \qquad \delta\circ d = d\circ\delta, \qquad d\circ d = 0. \quad (8.45)$$

Here the first equation is proved by the same argument as in Lemma 8.56, the second equation is obvious, and the third equation follows from Lemma 7.21.

The complex is equipped with a **double grading** by the integers k and p. The total grading is defined by

$$\deg(\omega) := k + p, \qquad \omega \in C^k(\mathscr{U}, \Omega^p),$$

and the degree-n part of the complex will be denoted by

$$\check{C}^n(\mathscr{U}) := \bigoplus_{k+p=n} C^k(\mathscr{U}, \Omega^p).$$

We write $\omega^{k,p}$ for the projection of $\omega \in \check{C}^s(\mathscr{U})$ onto $C^k(\mathscr{U}, \Omega^p)$. The double graded complex carries a boundary operator

$$D : \check{C}^n(\mathscr{U}) \to \check{C}^{n+1}(\mathscr{U})$$

defined by

$$(D\omega)^{k,p} := \delta\omega^{k-1,p} + (-1)^k d\omega^{k,p-1} \quad (8.46)$$

for $\omega \in \check{C}^n(\mathscr{U})$ and nonnegative integers k and p satisfying $k + p = n + 1$. The sign can be understood as arising from interchanging d and k.

Lemma 8.63. *The operator* (8.46) *satisfies* $D \circ D = 0$.

Proof. Let $\omega \in \check{C}^n(\mathscr{U})$ and choose k and p such that $k + p = n + 2$. Then

$$\begin{aligned} (D(D\omega))^{k,p} &= \delta(D\omega)^{k-1,p} + (-1)^k d(D\omega)^{k,p-1} \\ &= \delta\left(\delta\omega^{k-2,p} + (-1)^{k-1} d\omega^{k-1,p-1}\right) \\ &\quad +(-1)^k d\left(\delta\omega^{k-1,p-1} + (-1)^k d\omega^{k,p-2}\right) \\ &= \delta\delta\omega^{k-2,p} + (-1)^k(d\delta - \delta d)\omega^{k-1,p-1} + dd\omega^{k,p-2} \\ &= 0. \end{aligned}$$

The last equation follows from (8.45) and this proves the lemma. □

The complex $(\check{C}^*(\mathscr{U}), D)$ is called the **Čech–deRham complex** of the cover $\mathscr{U}$ and its cohomology

$$\check{H}^n(\mathscr{U}) := \frac{\ker\ D : \check{C}^n(\mathscr{U}) \to \check{C}^{n+1}(\mathscr{U})}{\mathrm{im}\ D : \check{C}^{n-1}(\mathscr{U}) \to \check{C}^n(\mathscr{U})}. \tag{8.47}$$

is called the **Čech–deRham cohomology** of $\mathscr{U}$. There are natural cochain homomorphisms

$$\begin{aligned} \iota &: C^k(\mathscr{U}, \mathbb{R}) \to C^k(\mathscr{U}, \Omega^0) \subset \check{C}^k(\mathscr{U}), \\ r &: \Omega^p(M) \to C^0(\mathscr{U}, \Omega^p) \subset \check{C}^p(\mathscr{U}). \end{aligned} \tag{8.48}$$

The operator ι is the inclusion of the constant functions and r is the restriction defined by $(r\omega)_i := \omega|_{U_i}$ for $i \in I$. The maps r, δ, ι, d are depicted in the following diagram. We will prove that all rows except for the first and all columns except for the first are exact in the case of a good cover.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \cdots . \\
\downarrow & & \downarrow r & & \downarrow r & & \downarrow r & & \\
C^0(\mathscr{U}, \mathbb{R}) & \xrightarrow{\iota} & C^0(\mathscr{U}, \Omega^0) & \xrightarrow{d} & C^0(\mathscr{U}, \Omega^1) & \xrightarrow{d} & C^0(\mathscr{U}, \Omega^1) & \xrightarrow{d} & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
C^1(\mathscr{U}, \mathbb{R}) & \xrightarrow{\iota} & C^1(\mathscr{U}, \Omega^0) & \xrightarrow{d} & C^1(\mathscr{U}, \Omega^1) & \xrightarrow{d} & C^1(\mathscr{U}, \Omega^1) & \xrightarrow{d} & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
C^2(\mathscr{U}, \mathbb{R}) & \xrightarrow{\iota} & C^2(\mathscr{U}, \Omega^0) & \xrightarrow{d} & C^2(\mathscr{U}, \Omega^1) & \xrightarrow{d} & C^2(\mathscr{U}, \Omega^1) & \xrightarrow{d} & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Lemma 8.64. *The sequence*

$$0 \to \Omega^p(M) \xrightarrow{r} C^0(\mathscr{U},\Omega^p) \xrightarrow{\delta} C^1(\mathscr{U},\Omega^p) \xrightarrow{\delta} C^2(\mathscr{U},\Omega^p) \xrightarrow{\delta} \cdots \tag{8.49}$$

is exact for every integer $p \geq 0$*. If* $\mathscr{U}$ *is a good cover of* M *then the sequence*

$$0 \to C^k(\mathscr{U},\mathbb{R}) \xrightarrow{\iota} C^k(\mathscr{U},\Omega^0) \xrightarrow{d} C^k(\mathscr{U},\Omega^1) \xrightarrow{d} C^k(\mathscr{U},\Omega^2) \xrightarrow{d} \cdots \tag{8.50}$$

is exact for every integer $k \geq 0$*.*

Proof. For the sequence (8.50) exactness follows immediately from Example 8.12 and the good cover condition. For the sequence (8.49) the good cover condition is not required. Exactness at $C^0(\mathscr{U},\Omega^p)$ follows directly from the definitions. To prove exactness at $C^k(\mathscr{U},\Omega^p)$ for $k \geq 1$ we choose a partition of unity $\rho_i : M \to [0,1]$ subordinate to the cover $\mathscr{U} = \{U_i\}_{i\in I}$. For $k \geq 1$ define the operator

$$h : C^k(\mathscr{U},\Omega^p) \to C^{k-1}(\mathscr{U},\Omega^p)$$

by

$$(h\omega)_{i_0\cdots i_{k-1}} := \sum_{i\in I} \rho_i \omega_{ii_0\cdots i_{k-1}} \tag{8.51}$$

for $\omega \in C^k(\mathscr{U},\Omega^p)$ and $(i_0,\dots,i_{k-1}) \in I_{k-1}(\mathscr{U})$, where each term in the sum is understood as the extension to the open set $U_{i_0} \cap \cdots \cap U_{i_k}$ by setting it equal to zero on the complement of $U_i \cap U_{i_0} \cap \cdots \cap U_{i_k}$. We prove that

$$\delta \circ h + h \circ \delta = \mathrm{id} : C^k(\mathscr{U},\Omega^p) \to C^k(\mathscr{U},\Omega^p) \tag{8.52}$$

for $k \geq 1$. This shows that if $\omega \in C^k(\mathscr{U},\Omega^p)$ satisfies $\delta\omega = 0$ then $\omega = \delta h\omega$ belongs to the image of δ. To prove (8.52) we compute

$$\begin{aligned}
(h\delta\omega)_{i_0\cdots i_k} &= \sum_{i\in I} \rho_i (\delta\omega)_{ii_0\cdots i_k} \\
&= \sum_{i\in I} \rho_i \left(\omega_{i_0\cdots i_k} - \sum_{\nu=0}^{k} (-1)^\nu \omega_{ii_0\cdots\widehat{i_\nu}\cdots i_k} \right) \\
&= \omega_{i_0\cdots i_k} - \sum_{\nu=0}^{k} (-1)^\nu \sum_{i\in I} \rho_i \omega_{ii_0\cdots\widehat{i_\nu}\cdots i_k} \\
&= \omega_{i_0\cdots i_k} - \sum_{\nu=0}^{k} (-1)^\nu (h\omega)_{i_0\cdots\widehat{i_\nu}\cdots i_k} \\
&= \big(\omega - \delta h\omega\big)_{i_0\cdots i_k}
\end{aligned}$$

for $\omega \in C^k(\mathscr{U},\Omega^p)$ and $(i_0,\dots,i_k) \in \mathcal{I}_k(\mathscr{U})$. This proves (8.52) and the lemma. □

Theorem 8.65. *Let $\mathscr{U}$ be a good cover of M. Then the homorphism*

$$r : \Omega^*(M) \to \check{C}^*(\mathscr{U}), \qquad \iota : C^*(\mathscr{U}, \mathbb{R}) \to \check{C}^*(\mathscr{U})$$

induce isomorphism

$$r^* : H^*(M) \to \check{H}^*(\mathscr{U}), \qquad \iota^* : H^*(\mathscr{U}, \mathbb{R}) \to \check{H}^*(\mathscr{U})$$

on cohomology.

Proof. We prove that r is injective in cohomology. Let $\omega \in \Omega^p(M)$ be closed and assume that $\omega^{0,p} := r\omega = (\omega|_{U_i})_{i\in I} \in C^0(\mathscr{U}, \Omega^p) \subset \check{C}^p(\mathscr{U})$ is exact. Then there are elements $\tau^{k-1,p-k} \in C^{k-1}(\mathscr{U}, \omega^{p-k})$, $k = 1, \dots, p$, such that $r\omega = D\tau$:

$$\begin{aligned} \omega^{0,p} &= d\tau^{0,p-1}, \\ 0 &= \delta\tau^{k-1,p-k} + (-1)^k d\tau^{k,p-k-1}, \qquad k = 1, \dots, p-1, \\ 0 &= \delta\tau^{p-1,0}. \end{aligned} \tag{8.53}$$

We must prove that ω is exact. To see this we observe that there are elements $\sigma^{k-2,p-k} \in C^{k-2}(\mathscr{U}, \Omega^{p-k})$, $p \geq k \geq 2$, satisfying

$$\begin{aligned} \delta\sigma^{p-2,0} &= \tau^{p-1,0}, \\ \delta\sigma^{k-2,p-k} &= \tau^{k-1,p-k} + (-1)^k d\sigma^{k-1,p-k-1}, \qquad p-1 \geq k \geq 2. \end{aligned} \tag{8.54}$$

The existence of $\sigma^{p-2,0}$ follows immediately from the last equation in (8.53) and Lemma 8.64. If $2 \leq k \leq p-1$ and $\sigma^{k-1,p-k-1}$ has been found such that

$$\delta\sigma^{k-1,p-k-1} = \tau^{k,p-k-1} + (-1)^{k+1} d\sigma^{k,p-k-2},$$

we have $d\delta\sigma^{k-1,p-k-1} = d\tau^{k,p-k-1}$ and hence

$$\delta\left(\tau^{k-1,p-k} + (-1)^k d\sigma^{k-1,p-k-1}\right) = \delta\tau^{k-1,p-k} + (-1)^k d\tau^{k,p-k-1} = 0.$$

Here the last equation follows from (8.53). Thus, by Lemma 8.64, there is an element $\sigma^{k-2,p-k}$ satisfying (8.54).

It follows from (8.53) with $k = 1$ that $\delta\tau^{0,p-1} = d\tau^{1,p-2}$ and from (8.54) with $k = 2$ that $\tau^{1,p-2} + d\sigma^{1,p-3} = \delta\sigma^{0,p-2}$. Hence

$$\begin{aligned} \delta\left(\tau^{0,p-1} - d\sigma^{0,p-2}\right) &= \delta\tau^{0,p-1} - d\tau^{1,p-2} = 0, \\ d\left(\tau^{0,p-1} - d\sigma^{0,p-2}\right) &= d\tau^{0,p-1} = \omega^{0,p}. \end{aligned} \tag{8.55}$$

The first equation in (8.55) shows that there is a global $(p-1)$-form $\widetilde{\tau}$ on M whose restriction to U_i agrees with the relevant component of the Čech–deRham cochain $\tau^{0,p-1} - d\sigma^{0,p-2} \in C^0(\mathscr{U}, \Omega^{p-1})$. The second equation in (8.55) shows that $d\widetilde{\tau} = \omega$. Hence ω is exact, as claimed.

We prove that r is surjective in cohomology. Let $\omega^{k,p-k} \in C^k(\mathscr{U}, \Omega^{p-k})$ be given for $k = 0, \dots, p$ and suppose that $D\omega = 0$:

$$\begin{aligned} 0 &= d\omega^{0,p}, \\ 0 &= \delta\omega^{k,p-k} + (-1)^{k+1} d\omega^{k+1,p-k-1}, \qquad k = 0, \dots, p-1, \\ 0 &= \delta\omega^{p,0}. \end{aligned} \tag{8.56}$$

We construct elements $\tau^{k-1,p-k} \in C^{k-1}(\mathscr{U}, \Omega^{p-k})$, $k = 1, \dots, p$, satisfying

$$\begin{aligned} \delta\tau^{p-1,0} &= \omega^{p,0}, \\ \delta\tau^{k-1,p-k} &= \omega^{k,p-k} + (-1)^{k+1} d\tau^{k,p-k-1}, \qquad k = 1, \dots, p-1. \end{aligned} \tag{8.57}$$

The existence of $\tau^{p-1,0}$ follows immediately from the last equation in (8.56) and Lemma 8.64. If $1 \le k \le p-1$ and $\tau^{k,p-k-1}$ has been found such that

$$\delta\tau^{k,p-k-1} = \omega^{k+1,p-k-1} + (-1)^{k+2} d\tau^{k+1,p-k-1},$$

we have $d\delta\tau^{k,p-k-1} = d\omega^{k+1,p-k-1}$ and hence

$$\delta\left(\omega^{k,p-k} + (-1)^{k+1} d\tau^{k,p-k-1}\right) = \delta\omega^{k,p-k} + (-1)^{k+1} d\omega^{k+1,p-k-1} = 0.$$

Here the last equation follows from (8.56). By exactness, this shows that there is an element $\tau^{k-1,p-k}$ satisfying (8.57). It follows from (8.57) that

$$\begin{aligned} (\omega - D\tau)^{0,p} &= \omega^{0,p} - d\tau^{0,p-1}, \\ (\omega - D\tau)^{k,p-k} &= \omega^{k,p-k} - \delta\tau^{k-1,p-k} - (-1)^k d\tau^{k,p-k-1} = 0, \\ (\omega - D\tau)^{p,0} &= \omega^{p,0} - \delta\tau^{p-1,0} = 0 \end{aligned} \tag{8.58}$$

for $k = 1, \dots, p-1$. Moreover, it follows from (8.56) with $k = 0$ that $\delta\omega^{0,p} = d\omega^{1,p-1}$ and from (8.57) with $k = 1$ that $\delta\tau^{0,p-1} = d\tau^{1,p-2}$. Hence

$$\begin{aligned} \delta(\omega - D\tau)^{0,p} &= \delta\left(\omega^{0,p} - d\tau^{0,p-1}\right) \\ &= d\left(\omega^{1,p-1} - \delta\tau^{0,p-1}\right) \\ &= d\left(-d\tau^{1,p-2}\right) \\ &= 0. \end{aligned}$$

This shows there is a global p-form $\widetilde{\omega}$ on M whose restriction to U_i agrees with the relevant component of $\omega^{0,p} - d\tau^{0,p-1} \in C^0(\mathscr{U}, \Omega^p)$. This form is closed and satisfies $r\widetilde{\omega} = \omega - D\tau$, by (8.58). Hence the cohomology class of ω in $\check{H}^p(\mathscr{U})$ belongs to the image of $r^* : H^p(M) \to \check{H}^p(\mathscr{U})$.

Thus we have proved that $r^* : H^*(M) \to \check{H}^*(\mathscr{U})$ is an isomorphism. The proof that $\iota^* : H^*(\mathscr{U}, \mathbb{R}) \to \check{H}^*(\mathscr{U})$ is an isomorphism as well follows by exactly the same argument with the rows and columns in our diagram interchanged. □

Proof of Theorem 8.60. Recall that the linear map

$$h : C^k(\mathscr{U},\Omega^p) \to C^{k-1}(\mathscr{U},\Omega^p)$$

in (8.51) has the form $(h\omega)_{i_0\cdots i_{k-1}} = \sum_{i\in I}\rho_i\omega_{ii_0\cdots i_{k-1}}$, and define the map

$$\Phi : C^k(\mathscr{U},\Omega^p) \to C^{k-1}(\mathscr{U},\Omega^{p+1})$$

by

$$(\Phi\omega)_{i_0\cdots i_{k-1}} := (-1)^k\sum_{i\in I} d\rho_i\wedge\omega_{ii_0\cdots i_{k-1}} = \sum_{i\in I} d\rho_i\wedge\omega_{i_0\cdots i_{k-1}i}$$

for $\omega\in C^k(\mathscr{U},\Omega^{p-k})$. The product with $d\rho_i$ guarantees that each summand on the right extends smoothly to $U_{i_0\cdots i_{k-1}}$ by setting it equal to zero on the complement of the intersection with U_i. These two operators satisfy

$$\mathrm{id} = \delta\circ h + h\circ\delta, \qquad -\Phi = \big((-1)^{k-1}d\big)\circ h + h\circ\big((-1)^k d\big)$$

on $C^k(\mathscr{U},\Omega^{p-k})$. Here the first equation is (8.52) and the second equation follows directly from the definitions. Combining these two equations we find

$$\mathrm{id} - \Phi = D\circ h + h\circ D.$$

Thus Φ induces the identity on $\check{H}^k(\mathscr{U})$.

Starting with $p = 0$ and iterating the operator k times we obtain a homomorphism

$$\Phi^k = \underbrace{\Phi\circ\Phi\circ\cdots\circ\Phi}_{k\text{ times}} : C^k(\mathscr{U},\Omega^0)\to C^0(\mathscr{U},\Omega^k),$$

inducing the identity on $\check{H}^k(\mathscr{U})$. This operator assigns to every element $f = (f_{i_0\cdots i_k})_{(i_0\cdots i_k)\in\mathcal{I}_k(\mathscr{U})}\in C^k(\mathscr{U},\Omega^0)$ the tuple $\Phi^k f\in C^0(\mathscr{U},\Omega^k)$ given by

$$(\Phi^k f)_i = \sum_{(i_1,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})} f_{ii_1\dots i_k}d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_k}\in\Omega^k(U_i).$$

Hence, by Remark 8.59, the following diagram commutes on the kernel of δ:

$$\begin{array}{ccccc}
C^k(\mathscr{U},\mathbb{R}) & \supset & \ker\delta & \xrightarrow{c\mapsto\omega_c} & \Omega^k(M) \\
 & & \downarrow\iota & & \downarrow r \\
 & & C^k(\mathscr{U},\Omega^0) & \xrightarrow{\Phi^k} & C^k(\mathscr{U},\Omega^k)
\end{array}.$$

Since Φ^k induces the identity on Čech–DeRham cohomology, we deduce that the composition of the homomorphism $H^k(\mathscr{U},\mathbb{R})\to H^*(M) : [c]\mapsto[\omega_c]$ in (8.42) with $r^* : H^*(M)\to\check{H}^k(\mathscr{U})$ is equal to $\iota^* : H^k(\mathscr{U},\mathbb{R})\to\check{H}^k(\mathscr{U})$. Hence it follows from Theorem 8.65 that the homomorphism (8.42) is an isomorphism. This proves the theorem. □

8.5.4 Product Structures

The Čech complex of an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ is equipped with a **cup product**. The definition of this product structure is quite straight forward, however, it requires the choice of an order relation $\prec$ on the index set I. Given such an ordering, each cochain

$$\omega = (\omega_{i_0\cdots i_k})_{(i_0,\ldots,i_k)\in\mathcal{I}_k(\mathscr{U})} \in C^k(\mathscr{U},\Omega^p)$$

is uniquely determined by the elements $\omega_{i_0\cdots i_k}$ for those tuples that satisfy $i_0 \prec i_1 \prec \cdots \prec i_k$. All the other elements are then determined by the equivariance condition under the action of the permutation group S_{k+1}.

Definition 8.66. *The* **cup product** *on $C^*(\mathscr{U},\Omega^*)$ is the bilinear map*

$$C^k(\mathscr{U},\Omega^p) \times C^\ell(\mathscr{U},\Omega^q) \to C^{k+\ell}(\mathscr{U},\Omega^{p+q}) : (\omega,\tau) \mapsto \omega \cup \tau$$

defined by

$$(\omega\cup\tau)_{i_0\cdots i_{k+\ell}} := (-1)^{\ell p}\omega_{i_0\cdots i_k} \wedge \tau_{i_k\cdots i_{k+\ell}} \tag{8.59}$$

for every $\omega \in \mathbb{C}^k(\mathscr{U},\Omega^p)$, every $\tau \in C^\ell(\mathscr{U},\Omega^q)$, and every $(k+\ell+1)$-tuple $(i_0, i_1, \ldots, i_{k+\ell}) \in \mathcal{I}_{k+\ell}(\mathscr{U})$ that satisfies

$$i_0 \prec i_1 \prec \cdots \prec i_{k+\ell}.$$

Here the right hand side in (8.59) *is understood as the restriction of the differential form to the open subset $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_{k+\ell}}$.*

Remark 8.67. The product structure on $C^*(\mathscr{U},\Omega^*)$ is sensitive to the choice of the ordering of the index set I and is not commutative in any way, shape, or form. In fact, the cup product $\tau \cup \omega$ associated to the reverse ordering agrees up to the usual sign $(-1)^{\deg(\omega)\deg(\tau)}$ with the cup product $\omega \cup \tau$ associated to the original ordering.

Remark 8.68. The sign in equation (8.59) is naturally associated to the interchanged indices p and ℓ.

Remark 8.69. The cup product on $C^*(\mathscr{U},\Omega^*)$ restricts to the product

$$(a\cup b)_{i_0\cdots i_{k+\ell}} = a_{i_0\cdots i_k} b_{i_k\cdots i_{k+\ell}}, \qquad i_0 \prec i_1 \prec \cdots \prec i_{k+\ell}, \tag{8.60}$$

on $C^*(\mathscr{U},\mathbb{R}) \subset C^*(\mathscr{U},\Omega^0)$.

Remark 8.70. The cup product on $C^*(\mathscr{U},\Omega^*)$ restricts to the exterior product for differential forms on $C^0(\mathscr{U},\Omega^*)$.

Lemma 8.71. *The cup product* (8.59) *on* $C^*(\mathscr{U},\Omega^*)$ *is associative and*

$$D(\omega\cup\tau) = (D\omega)\cup\tau + (-1)^{\deg(\omega)}\omega\cup(D\tau) \tag{8.61}$$

for $\omega\in C^k(\mathscr{U},\Omega^p)$ *and* $\tau\in C^\ell(\mathscr{U},\Omega^q)$, *where* $\deg(\omega)=k+p$.

Proof. The proof of associativity is left as an exercise. To prove (8.61) we compute

$$\begin{aligned}
\big(\delta(\omega\cup\tau)\big)_{i_0\cdots i_{k+\ell+1}} &= \sum_{\nu=0}^{k+\ell+1}(-1)^\nu(\omega\cup\tau)_{i_0\cdots\widehat{i_\nu}\cdots i_{k+\ell+1}} \\
&= \sum_{\nu=0}^{k}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\wedge\tau_{i_{k+1}\cdots i_{k+\ell+1}} \\
&\quad + \sum_{\nu=k+1}^{k+\ell+1}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots\widehat{i_\nu}\cdots i_{k+\ell+1}} \\
&= \sum_{\nu=0}^{k+1}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\wedge\tau_{i_{k+1}\cdots i_{k+\ell+1}} \\
&\quad + \sum_{\nu=k}^{k+\ell+1}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots\widehat{i_\nu}\cdots i_{k+\ell+1}} \\
&= (-1)^{\ell p}(\delta\omega)_{i_0\cdots i_{k+1}}\wedge\tau_{i_{k+1}\cdots i_{k+\ell+1}} \\
&\quad + (-1)^{\ell p+k}\omega_{i_0\cdots i_k}\wedge(\delta\tau)_{i_k\cdots i_{k+\ell+1}} \\
&= \big((\delta\omega)\cup\tau\big)_{i_0\cdots i_{k+\ell+1}} + (-1)^{k+p}\big(\omega\cup(\delta\tau)\big)_{i_0\cdots i_{k+\ell+1}}.
\end{aligned}$$

Thus we have proved that

$$\delta(\omega\cup\tau) = (\delta\omega)\cup\tau + (-1)^{\deg(\omega)}\omega\cup(\delta\tau). \tag{8.62}$$

Moreover,

$$\begin{aligned}
\big(d(\omega\cup\tau)\big)_{i_0\cdots i_{k+\ell+1}} &= (-1)^{\ell p}d\left(\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots i_{k+\ell}}\right) \\
&= (-1)^{\ell p}d\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots i_{k+\ell}} \\
&\quad + (-1)^{(\ell+1)p}\omega_{i_0\cdots i_k}\wedge d\tau_{i_k\cdots i_{k+\ell}}
\end{aligned}$$

Thus we have proved that

$$(-1)^{k+\ell}d(\omega\cup\tau) = \Big((-1)^k d\omega\Big)\cup\tau + (-1)^{\deg\omega}\omega\cup\Big((-1)^\ell d\tau\Big). \tag{8.63}$$

Equation (8.61) follows by taking the sum of equations (8.62) and (8.63). This proves the lemma. □

The cochain homomorphisms r and ι intertwine the product structures on the cochain level. Hence the induced homomorphisms on cohomology

$$r^* : H^*(M) \to \check{H}^*(\mathscr{U}), \qquad \iota^* : H^*(\mathscr{U}, \mathbb{R}) \to \check{H}^*(\mathscr{U})$$

also intertwine the product structures. If $\mathscr{U}$ is a good cover these are isomorphisms and hence, in this case, both cohomology groups $\check{H}^*(\mathscr{U})$ and $H^*(\mathscr{U}, \mathbb{R})$ inherit the commutativity properties of the cup product on deRham cohomology, although this is not at all obvious from the definitions.

8.5.5 DeRham's Theorem

There is a natural homomorphism

$$H^*_{\mathrm{dR}}(M) \to H^*_{\mathrm{sing}}(M, \mathbb{R}) \tag{8.64}$$

from the deRham cohomology of M to the singular cohomology with real coefficients, defined in terms of integration over smooth singular cycles. **DeRham's Theorem** asserts that this homomorphism is bijective. To prove this it suffices, in view of Theorem 8.60, to prove that the singular cohomology of M with real coefficients is isomorphic to the Čech cohomology $H^*(\mathscr{U}, \mathbb{R})$ associated to a good cover. The proof involves similar methods as that of Theorem 8.60 but will not be included in this manuscript. An excellent reference is the book of Bott and Tu [2].

Remark 8.72. Let M be a compact oriented smooth m-manifold without boundary. It is a deep theorem in algebraic topology that a suitable integer multiple of any integral singular homology class on M can be represented by a compact oriented submanifold without boundary, in the sense that any triangulation of the submanifold gives rise to a singular cycle representing the homology class. The details of this are outside the scope of the present manuscript. However, we mention without proof the following consequence of this result and DeRham's theorem:

There is a finite collection of compact oriented $(m - k_i)$-dimensional submanifolds without boundary

$$Q_i \subset M, \qquad i = 0, \ldots, n,$$

such that the cohomology classes of the closed forms

$$\tau_i = \tau_{Q_i} \in \Omega^{k_i}(M),$$

dual to the submanifolds as in Section 8.4.3, form a basis of $H^*(M)$.

Remark 8.73. *It follows from the assertion in Remark 8.72 that every closed form $\omega \in \Omega^k(M)$ that satisfies*

$$\int_P f^*\omega = 0$$

for every compact oriented smooth k-manifold P without boundary and every smooth map $f : P \to M$ is exact. (This implies that the homomorphism (8.64) *is injective.)*

For $k = 1$ this follows from Exercise 8.50. To see this in general, let Q_i and τ_i be chosen as in Remark 8.72 and denote by $I_k \subset \{0, \dots, n\}$ the set of all indices i such that

$$\dim Q_i = m - k_i = k, \qquad \deg(\tau_i) = k_i = m - k.$$

If $\omega \in \Omega^k(M)$ satisfies our assumptions then

$$\int_M \omega \wedge \tau_i = \int_{Q_i} \omega = 0$$

for every $i \in I_k$. Since the cohomology classes $[\tau_i]$ form a basis of $H^{m-k}(M)$ we have

$$\int_M \omega \wedge \tau = 0$$

for every closed $(m - k)$-form τ. Hence ω is exact, by Theorem 8.38.

Exercise 8.74. Define a homomorphism

$$H^1(M) \to \mathrm{Hom}(\pi_1(M, p_0), \mathbb{R}) : [\omega] \mapsto \rho_\omega \tag{8.65}$$

which assigns to every closed 1-form $\omega \in \Omega^1(M)$ the homomorphism

$$\rho_\omega : \pi_1(M, p_0) \to \mathbb{R}, \qquad \rho_\omega([\gamma]) := \int_{[0,1]} \gamma^*\omega,$$

for every smooth based loop $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1) = p_0$. By Theorem 8.1, ρ_ω depends only on the cohomology class of ω. By Exercise 8.50 the homomorphism $[\omega] \mapsto \rho_\omega$ is injective. Prove that it is surjective. **Hint:** Choose a good cover $\mathscr{U} = \{U_i\}_{i \in I}$ of M and, for each $i \in I$, choose a point $p_i \in U_i$ and a path $\gamma_i : [0, 1] \to M$ such that $\gamma_i(0) = p_0$ and $\gamma_i(1) = p_i$. For $(i, j) \in \mathcal{I}_1(\mathscr{U})$ define the number $c_{ij} \in \mathbb{R}$ by

$$c_{ij} := \rho(\gamma), \qquad \begin{cases} \gamma(t) = \gamma_i(4t), & \text{for } 0 \le t \le 1/4, \\ \gamma(t) \in U_i, & \text{for } 1/4 \le t \le 1/2, \\ \gamma(t) \in U_j, & \text{for } 1/2 \le t \le 3/4, \\ \gamma(t) = \gamma_j(4(1-t)), & \text{for } 3/4 \le t \le 1. \end{cases}$$

Prove that any two such paths γ are homotopic with fixed endpoints. Prove that the numbers c_{ij} determine a 1-cocycle in the Čech complex $C^1(\mathscr{U}, \mathbb{R})$. Prove that the 1-form

$$\omega_c := \sum_{(i,j)\in\mathcal{I}_1(\mathscr{U})} c_{ij}\rho_i d\rho_j$$

is closed and satisfies $\rho_{\omega_c} = \rho$. Note that the only conditions on $\mathscr{U}$, needed in this proof are that the sets U_i are connected and simply connected, and that each nonempty intersection $U_i \cap U_j$ is connected.

Exercise 8.75. Consider the circle $M = S^1$ with its standard counterclockwise orientation and let $S^1 = U_1 \cup U_2 \cup U_3$ be a good cover. Thus the sets U_1, U_2, U_3 are open intervals as are the intersections $U_1 \cap U_2$, $U_2 \cap U_3$, $U_3 \cap U_1$. Assume that in the counterclockwise ordering the endpoint of U_1 is contained in U_2 and the endpoint of U_2 in U_3. Prove that the composition of the isomorphism $H^1(\mathscr{U}, \mathbb{R}) \to H^1(S^1)$ with the isomorphism $H^1(S^1) \to \mathbb{R}$, given by integration, is the map

$$H^1(\mathscr{U}, \mathbb{R}) \to \mathbb{R} : [c_{23}, c_{13}, c_{12}] \mapsto c_{23} - c_{13} + c_{12}.$$

Deduce that the homomorphism $\rho_{\omega_c} : \pi_1(S^1) \to \mathbb{R}$ associated to a cycle $c \in C^1(\mathscr{U}, \mathbb{R})$ as in Exercise 8.74 maps the positive generator to the real number $c_{23} - c_{13} + c_{12}$.

Exercise 8.76. Choose a good cover $\mathscr{U}$ of the 2-sphere by four open hemispheres and compute its Čech complex. Find an explicit expression for the isomorphism $H^2(\mathscr{U}, \mathbb{R}) \to \mathbb{R}$ associated to the standard orientation.

Chapter 9

Vector Bundles and the Euler Class

In this chapter we introduce smooth vector bundles over smooth manifolds in the intrinsic setting. Basic definitions and examples are discussed in Section 9.1. In Section 9.2 we define *Integration over the Fiber*, introduce the *Thom Class*, prove the *Thom Isomorphism Theorem*, and relate the Thom class to intersection theory. In Section 9.3 we introduce the *Euler Class* of an oriented vector bundle and show that, if the rank of the bundle agrees with the dimension of the base and the base is oriented, its integral over the base, the *Euler Number*, is equal to the algebraic number of zeros of a section with only nondegenerate zeros. As an application we compute the product structure on the deRham cohomology of complex projective space.

9.1 Vector Bundles

In [16] we have introduced the notion of a vector bundle

$$\pi : E \to M$$

over an (embedded) manifold M as a subbundle of the product $M \times \mathbb{R}^\ell$ for some integer $\ell \geq 0$. In this section we show how to carry the definitions of vector bundles, sections, and vector bundle homomorphisms over to the intrinsic setting. This is also the appropriate framework for introducing structure groups of vector bundles. And we discuss the notion of orientability, which specializes to orientability of a manifold in the case of the tangent bundle.

9.1.1 Definitions and Remarks

Let M be a smooth m-manifold and n be a nonnegative integer. A **real vector bundle over M of rank n** consists of a smooth manifold E of dimension $m+n$, a smooth map

$$\pi : E \to M,$$

called the **projection**, a real vector space V of dimension n, an open cover $\{U_\alpha\}_{\alpha \in A}$ of M, a collection of diffeomorphisms

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V, \qquad \alpha \in A,$$

called **local trivializations**, that satisfy

$$\mathrm{pr}_1 \circ \psi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$$

so that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times V \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U_\alpha & \end{array} \tag{9.1}$$

commutes for every $\alpha \in A$, and a collection of smooth maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V), \qquad \alpha, \beta \in A,$$

called **transition maps**, that satisfy

$$\psi_\beta \circ \psi_\alpha^{-1}(p, v) = (p, g_{\beta\alpha}(p)v) \tag{9.2}$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$, and $v \in V$.

For $p \in M$ the set

$$E_p := \pi^{-1}(p)$$

is called the **fiber of E over p**. If

$$\mathrm{G} \subset \mathrm{GL}(V)$$

is a Lie subgroup and the transition maps $g_{\beta\alpha}$ all take values in G we call E a **vector bundle with structure group** G. We say that the structure group of a vector bundle E **can be reduced to** G if E can be covered by local trivializations whose transition maps all take values in G.

It is sometimes convenient to write an element of a vector bundle E as a pair (p, e) consisting of a point $p \in M$ and an element $e \in E_p$ of the fiber of E over p. This notation suggests that we may think of a vector bundle over M as a *functor* which assigns to each element $p \in M$ a vector space E_p. The definition in Section 9.1.1 then requires that the disjoint union of the vector spaces E_p is equipped with the structure of a smooth manifold whose coordinate charts are compatible with the projection π and with the vector space structures on the fibers.

Remark 9.1. If $\pi : E \to M$ is a vector bundle then the projection π is a surjective submersion because the diagram (9.1) commutes.

Remark 9.2. If $\pi : E \to M$ is a vector bundle then, for every $p \in M$, the fiber $E_p = \pi^{-1}(p)$ inherits a vector space structure from V via the bijection

$$\psi_\alpha(p) := \mathrm{pr}_2 \circ \psi_\alpha|_{E_p} : E_p \to V \tag{9.3}$$

for $\alpha \in A$ with $p \in U_\alpha$. In other words, for $\lambda \in \mathbb{R}$ and $e, e' \in E_p$ we define the sum $e + e' \in E_p$ and the product $\lambda e \in E_p$ by

$$e + e' := \psi_\alpha(p)^{-1}(\psi_\alpha(p)e + \psi_\alpha(p)e'), \qquad \lambda e := \psi_\alpha(p)^{-1}(\lambda\psi_\alpha(p)).$$

The vector space structure on E_p is independent of α because the map

$$\psi_\beta(p) \circ \psi_\alpha(p)^{-1} = g_{\beta\alpha}(p) : V \to V$$

is linear for all $\alpha, \beta \in A$ with $p \in U_\alpha \cap U_\beta$.

Remark 9.3. The transition maps of a vector bundle E satisfy the conditions

$$g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}, \qquad g_{\alpha\alpha} = \mathbb{1}, \tag{9.4}$$

for all $\alpha, \beta, \gamma \in A$. Here the first equation is understood on the intersection $U_\alpha \cap U_\beta \cap U_\gamma$ where all three transition maps are defined.

Conversely, every open cover $\{U_\alpha\}_{\alpha \in A}$ and every system of transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ satisfying (9.4) determines a vector bundle

$$\widetilde{E} := \bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times V / \sim$$

where the equivalence relation is given by

$$[\alpha, p, v] \sim [\beta, p, g_{\beta\alpha}(p)v]$$

for $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$, and $v \in V$. The projection $\pi : E \to M$ is given by $[\alpha, p, v] \mapsto p$ and the local trivializations are given by $[\alpha, p, v] \mapsto (p, v)$. These local trivializations satisfy (9.2). This vector bundle is isomorphic to E (see Section 9.1.4 below).

9.1.2 Examples and Exercises

Example 9.4 (Trivial Bundle). The simplest example of a vector bundle over M is the **trivial bundle**

$$E = M \times \mathbb{R}^n.$$

It has an obvious *global* trivialization. Every real rank-n vector bundle over M is locally isomorphic to the trivial bundle but there is not necessarily a global isomorphism. (See below for the definition of a vector bundle isomorphism.)

Example 9.5 (Möbius Strip). The simplest example of a nontrivial vector bundle is the real rank-1 vector bundle

$$E := \left\{(z,\zeta) \in S^1 \times \mathbb{C} \,|\, z^2\zeta \in \mathbb{R}\right\}$$

over the circle

$$S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\},$$

called the **Möbius strip**. **Exercise:** Prove that the Möbius strip does not admit a global trivialization; it does not admit a global nonzero section. (See below for the definition of a section.)

Example 9.6 (Tangent Bundle). Let M be a smooth m-manifold with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in A}$. The tangent bundle

$$TM := \{(p,v) \,|\, p \in M,\, v \in T_pM\}$$

is a vector bundle over M with the obvious projection $\pi : TM \to M$ and the local trivializations

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m, \qquad \psi_\alpha(p,v) := (p, d\phi_\alpha(p)v).$$

The transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(m,\mathbb{R})$ are given by

$$g_{\beta\alpha}(p) = d(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))$$

for $p \in U_\alpha \cap U_\beta$.

Exercise 9.7 (Dual bundle). Let $\pi : E \to M$ be a real vector bundle with local trivializations $\psi_\alpha(p) : E_p \to V$. Show that the **dual bundle**

$$E^* := \{(p,e^*) \,|\, p \in M,\, e^* \in \mathrm{Hom}(E_p,\mathbb{R})\}$$

is a vector bundle with V replaced by V^* in the local trivializations and that the transition maps are related by $g_{\beta\alpha}^{E^*} = (g_{\alpha\beta}^E)^* : U_\alpha \cap U_\beta \to \mathrm{GL}(V^*)$. Deduce that the **cotangent bundle** T^*M is a vector bundle over M.

Example 9.8 (Exterior power). The **kth exterior power**

$$\Lambda^k T^*M := \left\{(p,\omega)\,|\,p \in M,\ \omega \in \Lambda^k T_p^*M\right\}$$

of the cotangent bundle is a real vector bundle with the the local trivializations given by pushforward under the derivatives of the coordinate charts:

$$(d\phi_\alpha(p)^{-1})^* : \Lambda^k T_p^*M \to \Lambda_k(\mathbb{R}^m)^*.$$

The transition maps of $\Lambda^k T^*M$ are then given by

$$g_{\beta\alpha}^{\Lambda^k T^*M}(p) = (d(\phi_\alpha \circ \phi_\beta^{-1})(\phi_\beta(p)))^* \in \mathrm{GL}(\Lambda^k(\mathbb{R}^m)^*)$$

for $p \in U_\alpha \cap U_\beta$.

Example 9.9 (Pullback). Let $\pi^E : E \to M$ be a real vector bundle with local trivializations $\psi_\alpha^E(p) : E_p \to V$ and let $f : N \to M$ be a smooth map. Then the **pullback bundle**

$$f^*E := \left\{(q,e)\,|\,q \in N,\ e \in E,\ \pi^E(e) = f(q)\right\} \subset N \times E$$

is a submanifold of $N \times E$ and a vector bundle over N with the obvious projection $\pi^{f^*E} : f^*E \to N$ onto the first factor, the local trivializations $\psi_\alpha^{f^*E}(q) = \psi_\alpha^E(f(q)) : (f^*E)_q = E_{f(q)} \to V$ for $q \in f^{-1}(U_\alpha)$ and the transition maps

$$g_{\beta\alpha}^{f^*E} = g_{\beta\alpha}^E \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to \mathrm{GL}(V).$$

Example 9.10 (Whitney Sum). Let $\pi^E : E \to M$, $\pi^F : F \to M$ be vector bundles with local trivializations $\psi_\alpha^E(p) : E_p \to V$, $\psi_\alpha^F(p) : F_p \to V$ for $p \in U_\alpha$ (over the same open cover). The **Whitney sum**

$$E \oplus F := \bigcup_{p \in M} \{p\} \times (E_p \oplus F_p),$$

is a vector bundle over M with the obvious projection $\pi : E \oplus F \to M$, the local trivializations

$$\psi_\alpha^{E\oplus F}(p) := \psi_\alpha^E(p) \oplus \psi_\alpha^F(p) : E_p \oplus F_p \to V \oplus W, \qquad p \in U_\alpha,$$

and the transition maps

$$g_{\beta\alpha}^{E\oplus F} = g_{\beta\alpha}^E \oplus g_{\beta\alpha}^F : U_\alpha \cap U_\beta \to \mathrm{GL}(V \oplus W).$$

Replacing everywhere $\oplus$ by $\otimes$ we obtain the **tensor product** of E and F.

Exercise 9.11 (Normal Bundle). Let M be a smooth m-manifold and $Q \subset M$ be a k-dimensional submanifold. Choose a Riemannian metric on M. Prove that the **normal bundle**

$$TQ^\perp := \{(p, v) \,|\, p \in Q,\, v \in T_pM,\, v \perp T_qQ\}$$

is a smooth vector bundle over Q of rank $m-k$. **Hint:** If Q is totally geodesic one can use the Levi–Civita connection on M to construct local trivialization of the normal bundle. Alternatively, one can use geodesics to find coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ such that $\phi_\alpha(U_\alpha \cap Q) = \phi_\alpha(U_\alpha) \cap (\mathbb{R}^k \times \{0\})$ and $v \perp T_qQ$ if and only if $d\phi_\alpha(q)v \in \{0\} \times \mathbb{R}^{m-k}$ for all $q \in Q$ and $v \in T_qM$. Yet another method is to identitfy the normal bundle with the quotient bundle $TM|_Q/TQ$ and use an arbitrary submanifold chart to find a local trivialization modelled on the quotient space $V = \mathbb{R}^m/\mathbb{R}^k$.

9.1.3 Sections

Let $\pi : E \to M$ be a real vector bundle over a smooth manifold. A **section of** E is a smooth map $s : M \to E$ such that

$$\pi \circ s = \mathrm{id} : M \to M.$$

The set of sections of E is a real vector space, denoted by

$$\Omega^0(M, E) := \{s : M \to E \,|\, s \text{ is smooth and } \pi \circ s = \mathrm{id}\}.$$

If we write a point in E as a pair (p, e) with $p \in M$ and $e \in E_p$, then we can think of a section of E as a natural transformation which assigns to each element p of M and element $s(p)$ of the vector space E_p such that the map $M \to E : p \mapsto (p, s(p))$ is smooth. Slightly abusing notation we will switch between these two points of view whenever convenient and use the same letter s for the map $M \to E : p \mapsto (p, s(p))$ and for the assignment $p \mapsto s(p) \in E_p$.

Remark 9.12. In the local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ a section $s : M \to E$ is given by smooth maps $s_\alpha : U_\alpha \to V$ such that

$$\psi_\alpha(s(p)) := (p, s_\alpha(p)). \tag{9.5}$$

These maps satisfy the condition

$$s_\beta = g_{\beta\alpha} s_\alpha \tag{9.6}$$

on $U_\alpha \cap U_\beta$. Conversely, every collection of smooth maps $s_\alpha : U_\alpha \to V$ satisfying (9.6) determine a unique global section $s : M \to E$ via (9.5).

Example 9.13 (**Zero Section).** The zero section

$$\iota : M \to E, \qquad \iota(p) := 0_p \in E_p,$$

assigns to each $p \in M$ the zero element of the fiber $E_p = \pi^{-1}(E)$ with respect to the vector space structure of Remark 9.2. Its image is a submanifold

$$Z := \iota(M) = \{0_p \,|\, p \in M\} \subset E,$$

which will also be called the **zero section of** E.

Exercise 9.14. For every vector bundle $\pi : E \to M$, every $p \in M$, and every $e \in E_p$, there is a smooth section $s : M \to E$ such that $s(p) = e$.

Example 9.15. The space of sections of the tangent bundle is the space of vector fields, the space of sections of the cotangent bundle is the space of 1-forms, and the space of sections of the kth exterior power of the cotangent bundle is the space of k-forms on M:

$$\Omega^0(M, TM) = \mathrm{Vect}(M), \qquad \Omega^0(M, \Lambda^k T^*M) = \Omega^k(M).$$

If $Q \subset M$ is a submanifold of a Riemannian manifold then the space of sections of the normal bundle of Q is the space $\Omega^0(Q, TQ^\perp) = \mathrm{Vect}^\perp(Q)$ of normal vector fields along Q.

9.1.4 Vector Bundle Homomorphisms

Let $\pi^E : E \to M$ and $\pi^F : F \to M$ be real vector bundles A **vector bundle homomorphism** is a smooth map $\Phi : E \to F$ such that

$$\pi^F \circ \Phi = \pi^E$$

and, for every $p \in M$, the restriction

$$\Phi_p := \Phi|_{E_p} : E_p \to F_p$$

is a linear map. A **vector bundle isomorphism** is a bijective vector bundle homomorphism. The vector bundles E and F are called **isomorphic** if there exists a vector bundle isomorpism $\Phi : E \to F$.

Exercise 9.16. (i) Every vector bundle isomorphism is a diffeomorphism.

(ii) Every injective vector bundle homomorphism is an embedding.

(iii) Every real vector bundle over a compact manifold M admits an injective vector bundle homomorphism $\Phi : E \to M \times \mathbb{R}^N$ for some integer N. **Hint:** Use a finite collection of local trivializations and a partition of unity.

Exercise 9.17. The Möbius strip $\pi : E \to S^1$ in Example 9.5 is not isomorphic to the trivial bundle $F := S^1 \times \mathbb{R}$. The tangent bundle TM of any manifold M is isomorphic to the cotangent bundle T^*M.

Exercise 9.18. The set

$$\mathrm{Hom}(E, F) := \bigcup_{p \in M} \{p\} \times \mathrm{Hom}(E_p, F_p)$$

is a vector bundle over M and the space of smooth sections of $\mathrm{Hom}(E, F)$ is the space of vector bundle homomorphisms from E to F. The vector bundle $E^* \otimes F$ is isomorphic to $\mathrm{Hom}(E, F)$.

9.1.5 Orientation

A vector bundle $\pi : E \to M$ is called **orientable** if its local trivializations can be chosen such that the transition maps take values in the group $\mathrm{GL}^+(V)$ of orientation preserving automorphisms of V, i.e. for all $\alpha, \beta \in A$ we have

$$g_{\beta\alpha}(p) = \psi_\beta(p) \circ \psi_\alpha(p)^{-1} \in \mathrm{GL}^+(V), \qquad p \in U_\alpha \cap U_\beta. \tag{9.7}$$

It is called **oriented** if V is oriented and (9.7) holds.

A vector bundle $\pi : E \to M$ is orientable if and only if its structure group can be reduced to $\mathrm{GL}^+(V)$. Care must be taken to distinguish between the orientability of E as a vector bundle and the orientability of E as a manifold. By definition, a manifold M is orientable if and only if its tangent bundle is orientable as a vector bundle. Thus E is orientable as a manifold if and only if its tangent bundle TE is orientable as a vector bundle. For example the trivial bundle $E = M \times \mathbb{R}^n$ is always orientable as a vector bundle but the manifold $M \times \mathbb{R}^n$ is only orientable if M is. Conversely, the tangent bundle of any manifold, orientable or not, is always an orientable manifold in the sense that its tangent bundle TTM is an orientable vector bundle.

Exercise 9.19. Let M be an orientable manifold and $\pi : E \to M$ be a real vector bundle. Then E is orientable as a vector bundle if and only if the manifold E is orientable.

Exercise 9.20. The Möbius strip in Example 9.5 is not orientable.

Exercise 9.21. A vector bundle $\pi : E \to M$ of rank n is oriented if and only the fibers E_p are equipped with orientations that fit together smoothly in the following sense: for every $p_0 \in M$ there is an open neighborhood $U \subset M$ of p_0 and there are sections $s_1, \dots, s_n : U \to E$ over U such that the vectors $s_1(p), \dots, s_n(p)$ form a positive basis of E_p for every $p \in U$.

Exercise 9.22. The tangent bundle of the tangent bundle is orientable.

9.2 The Thom Class

We assume throughout this section that M is a compact oriented smooth m-manifold without boundary and $\pi : E \to M$ is an oriented real vector bundle of rank n. As M is oriented, no ambiguity can arise as to the meaning of the term *"oriented"* for the vector bundle E; it is oriented both as a manifold and as a vector bundle.

9.2.1 Integration over the Fiber

Definition 9.23. *For $k = 0, 1, \dots, m$ we define a linear map*

$$\pi_* : \Omega_c^{n+k}(E) \to \Omega^k(M).$$

Let $\tau \in \Omega_c^{n+k}(E)$ be given and choose $p \in M$ and $v_1, \dots, v_k \in T_pM$. Associated to these data is a differential form

$$\tau^{p,v_1,\dots,v_k} \in \Omega_c^n(E_p),$$

defined as follows. Given $e \in E_p = \pi^{-1}(p)$ and tangent vectors

$$e_1, \dots, e_n \in T_eE_p = \ker\, d\pi(e) = E_p,$$

choose lifts $\widetilde{v}_i \in T_eE_p$ such that

$$d\pi(e)\widetilde{v}_i = v_i, \qquad i = 1, \dots, k,$$

and define

$$(\tau^{p,v_1,\dots,v_k})_e(e_1, \dots, e_n) := \tau_e(\widetilde{v}_1, \dots, \widetilde{v}_k, e_1, \dots, e_n). \tag{9.8}$$

The expression on the right is independent of the choice of the lifts $\widetilde{v}_i$; namely, if the e_j are linearly independent any two choices of lifts $\widetilde{v}_i$ differ by a linear combination of the e_j, and if the e_j are linearly dependent the right hand side of (9.8) *vanishes for any choice of the $\widetilde{v}_i$. Now the pushforward $\pi_*\tau \in \Omega^k(M)$ is defined by*

$$(\pi_*\tau)_p(v_1, \dots, v_k) := \int_{E_p} \tau^{p,v_1,\dots,v_k} \tag{9.9}$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$. The integral is well defined because $\tau^{p,v_1,\dots,v_k}$ has compact support and E_p is an oriented n-dimensional manifold.

Exercise 9.24. Prove that the map

$$(\pi_*\tau)_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R}$$

in (9.9) is an alternating k-form for every p and that these alternating k-forms fit together smoothly. Prove that the map $\tau \mapsto \pi_*\tau$ is linear.

Example 9.25. If $\tau \in \Omega^n_c(E)$ then $\pi_*\tau \in \Omega^0(M)$ is the smooth real valued function on M defined by

$$(\pi_*\tau)(p) = \int_{E_p} \tau$$

for $p \in M$.

Example 9.26. The map $\pi_* : H^{k+1}_c(M \times \mathbb{R}) \to H^k_c(M)$ in the proof of Lemma 8.31 is an example of integration over the fiber.

Lemma 9.27. *Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary and let $\pi_* : \Omega^{n+*}_c(E) \to \Omega^*(M)$ be the operator of Definition 9.23. Then*

$$\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \pi_*\tau. \tag{9.10}$$

for every $\omega \in \Omega^\ell(M)$ and every $\tau \in \Omega^{n+k}_c(E)$. Moreover,

$$\pi_* \circ d^E = d^M \circ \pi_* \tag{9.11}$$

and

$$\int_M \omega \wedge \pi_*\tau = \int_E \pi^*\omega \wedge \tau \tag{9.12}$$

for every $\omega \in \Omega^{m-k}(M)$ and every $\tau \in \Omega^{n+k}_c(E)$.

Proof. The proof of equation (9.10) relies on the observation that the vectors $e_i \in T_eE_p = E_p$, used in the definition of the compactly supported n-form $(\pi^*\omega \wedge \tau)^{p,v_1,\dots,v_{k+\ell}}$ on E_p in Definition 9.23, can only lead to nonzero terms when they appear in τ. The details are left as an exercise for the reader.

To prove (9.11) we will work in a local trivialization of E followed by local coordinates on M. Thus we consider the vector bundle

$$U \times \mathbb{R}^n$$

over an open set $U \subset \mathbb{R}^m$. We use coordinates $x^1, \dots, x^m$ on U and $t^1, \dots, t^n$ on $\mathbb{R}^n$. Thus our $(n+k)$-form $\tau \in \Omega^{n+k}(U \times \mathbb{R}^n)$ can be written in the form

$$\tau = \sum_{|J|+|K|=n+k} \tau_{J,K}(x,t)\, dx^J \wedge dt^K. \tag{9.13}$$

The compact support condition now translates into the assumption that the support of τ is contained in the product of U with a compact subset of $\mathbb{R}^n$. (Such forms are said to have **vertical compact support**; see [2].)

Integration over the fiber yields a k-form $\pi_*\tau \in \Omega^k(U)$ given by

$$\pi_*\tau = \sum_{|J|=k} \left(\int_{\mathbb{R}^n} \tau_{J,K_n}(x,t)\, dt^1 \cdots dt^n \right) dx^J, \tag{9.14}$$

where K_n denotes the maximal multi-index $K_n = (1,\dots,n)$. Next we apply the same operation to the form

$$\begin{aligned} d\tau = &\sum_{|J|+|K|=n+k} \sum_{i=1}^m \frac{\partial \tau_{J,K}}{\partial x^i}(x,t)\, dx^i \wedge dx^J \wedge dt^K \\ &+ \sum_{|J|+|K|=n+k} \sum_{j=1}^n \frac{\partial \tau_{J,K}}{\partial t^j}(x,t)\, dt^j \wedge dx^J \wedge dt^K. \end{aligned}$$

The key observation is that, for every fixed element $x \in U$, the second summand belongs to the image of the operator $d : \Omega_c^{n-1}(\mathbb{R}^n) \to \Omega_c^n(\mathbb{R}^n)$ and hence its integral over $\mathbb{R}^n$ vanishes by Stokes' Theorem 7.26. Thus integration over the fiber yields the $(k+1)$-form

$$\begin{aligned} \pi_* d\tau &= \sum_{|J|=k} \sum_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{\partial \tau_{J,K_n}}{\partial x^i}(x,t)\, dt^1 \cdots dt^n \right) dx^i \wedge dx^J \\ &= \sum_{i=1}^m \sum_{|J|=k} \left(\frac{\partial}{\partial x^i} \int_{\mathbb{R}^n} \tau_{J,K_n}(x,t)\, dt^1 \cdots dt^n \right) dx^i \wedge dx^J \\ &= d\pi_*\tau. \end{aligned}$$

Here the second equation follows by interchanging differentiation and integration and the last equation follows from (9.14). This proves (9.11).

We prove (9.12). Using a partition of unity on M we may again reduce the identity to a computation in local coordinates. Thus we assume that $\tau \in \Omega^{n+k}(U \times \mathbb{R}^n)$ is given by (9.13) and has vertical compact support as before, and that $\omega \in \Omega_c^{m-k}(U)$ has the form

$$\omega = \sum_{|I|=m-k} \omega_I(x)\, dx^I$$

Then both forms $\pi^*\omega \wedge \tau \in \Omega_c^{m+n}(U \times \mathbb{R}^n)$ and $\omega \wedge \pi_*\tau \in \Omega_c^m(U)$ have compact support. To compare their integral it is convenient to define a number $\varepsilon(I,J) \in \{\pm 1\}$ by

$$dx^I \wedge dx^J =: \varepsilon(I,J) dx^1 \wedge \cdots \wedge dx^m$$

for multi-indices I and J with $|I| = m-k$ and $|J| = k$.

With this setup we obtain

$$\begin{aligned}
&\int_U \omega \wedge \pi_*\tau \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \int_U \omega_I(x) \left(\int_{\mathbb{R}^n} \tau_{J,K_n}(x,t) dt^1 \cdots dt^n \right) dx^I \wedge dx^J \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \varepsilon(I,J) \int_U \int_{\mathbb{R}^n} \omega_I(x) \tau_{J,K_n}(x,t) dt^1 \cdots dt^n dx^1 \cdots dx^m \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \varepsilon(I,J) \int_{U\times\mathbb{R}^n} \omega_I(x) \tau_{J,K_n}(x,t) dx^1 \cdots dx^m dt^1 \cdots dt^n \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \int_{U\times\mathbb{R}^n} \omega_I(x) \tau_{J,K_n}(x,t) dx^I \wedge dx^J \wedge dt^{K_n} \\
&= \int_{U\times\mathbb{R}^n} \pi^*\omega \wedge \tau.
\end{aligned}$$

Here the third equality follows from Fubini's thoerem. This proves (9.12) and the lemma. □

9.2.2 Thom Forms

Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary. A closed compactly supported n-form $\tau \in \Omega_c^n(E)$ is called a **Thom form on** E if

$$\pi_*\tau = 1.$$

The next two lemmas characterize Thom forms and establish their existence.

Lemma 9.28. *Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary. Denote by $\iota : M \to E$ the zero section, let $\lambda \in \mathbb{R}$, and let $\tau \in \Omega_c^n(E)$ be closed. Then the following are equivalent.*

(a) $\pi_*\tau = \lambda$.

(b) *Every m-form $\omega \in \Omega^m(M)$ satisfies*

$$\int_E \pi^*\omega \wedge \tau = \lambda \int_M \omega.$$

(c) *Every closed m-form $\sigma \in \Omega^m(E)$ satisfies*

$$\int_E \sigma \wedge \tau = \lambda \int_M \iota^*\sigma.$$

Proof. We prove that (a) is equivalent to (b). By Lemma 9.27 we have

$$\int_M \omega \wedge \pi_* \tau = \int_E \pi^* \omega \wedge \tau$$

for every $\omega \in \Omega^m(M)$. Condition (a) holds if the term on the left is equal to $\lambda \int_M \omega$ for every ω and (b) holds if the term on the right is equal to $\lambda \int_M \omega$ for every ω. Thus (a) is equivalent to (b).

We prove that (b) is equivalent to (c). Since $\pi \circ \iota = \mathrm{id}$ we have

$$\iota^* \pi^* \omega = (\pi \circ \iota)^* \omega = \omega$$

for every $\omega \in \Omega^m(M)$. Hence (b) follows directly from (c) with $\sigma := \iota^* \omega$. Conversely, assume (b) and let $\sigma \in \Omega^m(E)$ be a closed m-form. Since the map $\iota \circ \pi : E \to E$ is the projection onto the zero section, it is homotopic to the identity via the homotopy $f_t : E \to E$, given by $f_t(e) = te$, with

$$f_0 = \iota \circ \pi, \qquad f_1 = \mathrm{id}.$$

Hence it follows from Theorem 8.1 that the m-form

$$\sigma - \pi^* \iota^* \sigma \in \Omega^m(E)$$

is exact for every closed form $\sigma \in \Omega^m(E)$. This implies

$$\int_E \sigma \wedge \tau = \int_E \pi^* \iota^* \sigma \wedge \tau = \lambda \int_M \iota^* \sigma.$$

Here the last equation follows from (b). This proves the lemma. □

Remark 9.29. A subset $U \subset E$ of a vector bundle is called **star shaped** if it intersects each fiber of E in a star shaped set centered at zero:

$$e \in U, \quad 0 \le t \le 1 \qquad \Longrightarrow \qquad te \in U.$$

The proof of Lemma 9.28 shows that, if $U \subset E$ is a star shaped open neighborhood of the zero section and $\tau \in \Omega^n_c(E)$ satisfies

$$\mathrm{supp}(\tau) \subset U, \qquad d\tau = 0, \qquad \pi_* \tau = 1,$$

then (c) continues to hold for every closed m-form $\sigma \in \Omega^m(U)$. Namely, in this case the m-form $f_t^* \sigma$, with $f_t(e) = te$, is defined on all of U for $0 \le t \le 1$ and hence

$$\sigma - \pi^* \iota^* \sigma = f_1^* \sigma - f_0^* \sigma$$

is an exact m-form on U, by Theorem 8.1. Hence the integral of its exterior product with τ vanishes, by Stokes' theorem.

Lemma 9.30. *Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary.*
(i) *For every $\beta \in \Omega_c^{n-1}(E)$ we have*

$$\pi_* d\beta = 0.$$

(ii) *For every open neighborhood $U \subset E$ of the zero section there is a compactly supported m-form $\tau \in \Omega_c^n(E)$ such that*

$$\mathrm{supp}(\tau) \subset U, \qquad d\tau = 0, \qquad \pi_*\tau = 1.$$

(iii) *If $\tau_0, \tau_1 \in \Omega_c^n(E)$ satisfy*

$$d\tau_0 = d\tau_1 = 0, \qquad \pi_*\tau_0 = \pi_*\tau_1 = 1,$$

then there is a compactly supported $(n-1)$-form $\beta \in \Omega_c^{n-1}(E)$ such that

$$\tau_1 - \tau_0 = d\beta.$$

Proof. We prove (i). Since $\partial E = \emptyset$ it follows from Stokes' Theorem that

$$\int_E \pi^*\omega \wedge d\beta = \int_E d(\pi^*\omega \wedge \beta) = 0$$

for every $\omega \in \Omega^m(M)$. Hence

$$\pi_* d\beta = 0,$$

by Lemma 9.28. This proves (i).

We prove (ii). Let $\iota : M \to E$ be the inclusion of the zero section as in Example 9.13 and define the linear operator $T : H^m(E) \to \mathbb{R}$ by

$$T([\sigma]) := \int_M \iota^*\sigma$$

for every closed m-form $\sigma \in \Omega^m(E)$. Since E is an oriented manifold and has a good cover it satisfies Poincaré duality, by Theorem 8.38. Hence there is a compactly supported closed n-form $\tau \in \Omega_c^n(E)$ such that

$$\sigma \in \Omega^m(E), \quad d\sigma = 0 \qquad \Longrightarrow \qquad \int_E \sigma \wedge \tau = T([\sigma]) = \int_M \iota^*\sigma.$$

By Lemma 9.28 with $\lambda = 1$, this implies

$$\pi_*\tau = 1.$$

Since τ has compact support there is a constant $\lambda \geq 1$ such that

$$e \in \operatorname{supp}(\tau) \qquad \Longrightarrow \qquad \lambda^{-1} e \in U.$$

Consider the homotopy $f_t : E \to E$ defined by

$$f_t(e) := te, \qquad 1 \leq t \leq \lambda.$$

It is proper and satisfies

$$\operatorname{supp}(f_\lambda^* \tau) = f_\lambda^{-1}(\operatorname{supp}(\tau)) \subset U.$$

Hence, by Remark 8.26, there is a $\beta \in \Omega_c^{n-1}(E)$ such that

$$f_\lambda^* \tau - \tau = d\beta.$$

The n-form $f_\lambda^* \tau \in \Omega_c^n(E)$ is closed and supported in U. Moreover, by (i), it satisfies

$$\pi_*(f_\lambda^* \tau) = 1.$$

This proves (ii).

We prove (iii). By assumption $\pi_*(\tau_1 - \tau_0) = 0$ and $\tau_1 - \tau_0$ is closed. Hence it follows from Lemma 9.28 with $\lambda = 0$ that

$$\int_E \sigma \wedge (\tau_1 - \tau_0) = 0$$

for every closed m-form $\sigma \in \Omega^m(E)$. Hence (iii) follows from Poincaré duality in Theorem 8.38. This proves the lemma. □

Remark 9.31. Lemma 9.30 remains valid when M is not orientable. However, the proof cannot use Poincaré duality. The result can then be obtained by using the Mayer–Vietoris sequence directly and this requires an extension of the theory to noncompact base manifolds M.

Remark 9.32. Lemma 9.30 remains valid for noncompact base manifolds with appropriate modifications. The required modification involves differential forms on E with socalled *vertical compact support* (see Bott–Tu [2]). If M is oriented the proof remains essentially the same.

9.2.3 The Thom Isomorphism Theorem

Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary. By Lemma 9.30 there is a Thom form on E and its cohomology class is independent of the choice of the Thom form. It is called the **Thom class of** E and will be denoted by

$$\tau(E) := [\tau] \in H_c^n(E), \qquad \tau \in \Omega_c^n(E), \qquad d\tau = 0, \qquad \pi_*\tau = 1. \tag{9.15}$$

Theorem 9.33 (Thom Isomorphism Theorem). *Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary. Then the homomorphism*

$$\pi_* : H_c^{n+k}(E) \to H^k(M) \tag{9.16}$$

is an isomorphism for $k = 0, 1, \dots, m$ and its inverse is the homomorphism

$$H^k(M) \to H_c^{n+k}(E) : a \mapsto \pi^* a \cup \tau(E). \tag{9.17}$$

Moreover $H_c^k(E) = 0$ for $k < n$.

Proof. Both manifolds M and E are oriented and have finite good covers and therefore satisfy Poincaré duality. Thus

$$H_c^{n+k}(E) \cong H^{m-k}(E) \cong H^{m-k}(M) \cong H^k(M)$$

for every integer k. Here the first and last isomorphisms exist by Poincaré duality (Theorem 8.38) and the second isomorphism exists by Corollary 8.5, because E is homotopy equivalent to M: the projection $\pi : E \to M$ is a homotopy equivalence and the inclusion of the zero section $\iota : M \to E$ is a homotopy inverse. In particular, for $k < 0$ the last two terms above are zero so $H_c^*(E)$ vanishes in degrees less than n.

Now let $T : H^k(M) \to H_c^{n+k}(E)$ be the homomorphisms (9.17). Then

$$T[\omega] = [\pi^*\omega \wedge \tau]$$

for every closed k-form $\omega \in \Omega^k(M)$, where $\tau \in \Omega_c^n(E)$ is a Thom form. Hence, by equation (9.10) in Lemma 9.27, we have

$$\pi_* T[\omega] = [\pi_*(\pi^*\omega \wedge \tau)] = [\omega \wedge \pi_*\tau] = [\omega]$$

for every closed k-form $\omega \in \Omega^k(M)$. Thus $\pi_* \circ T = \mathrm{id} : H^k(M) \to H^k(M)$. Since $H^k(M)$ and $H_c^{n+k}(M)$ have the same dimension it follows that π_* is an isomorphism with inverse T. This proves the theorem. □

Exercise 9.34 (Pullback). Let $\pi : E \to M$ and $\pi' : M' \to E'$ be oriented real rank-n vector bundles over compact oriented smooth manifolds without boundary. Let $\phi : M' \to M$ and $\Phi : E' \to E$ be smooth maps such that $\pi' \circ \Phi = \phi \circ \pi$ and the map $\Phi_p := \Phi|_{E_p} : E_p \to E'_{\phi(p)}$ is an orientation preserving vector space isomorphism for every $p \in M$. Prove that

$$\Phi^*\tau(E) = \tau(E') \in H_c^n(E').$$

9.2.4 Intersection Theory Revisited

It is interesting to review intersection theory in the light of the above results on the Thom class. We consider the following setting. Let M be an oriented (not necessarily compact) m-dimensional manifold without boundary and

$$Q \subset M$$

be a compact oriented $(m-n)$-dimensional submanifold without boundary. We also fix a Riemannian metric on M. For $\varepsilon > 0$ sufficiently small we consider the ε-neighborhood $TQ_\varepsilon^\perp$ of the zero section in the normal bundle and the tubular ε-neighborhood $U_\varepsilon \subset M$ of Q. These sets are defined by

$$\begin{aligned} TQ_\varepsilon^\perp &:= \left\{ (q,v) \,\middle|\, \begin{array}{l} q \in Q,\, v \in T_qM, \\ v \perp T_qQ,\, |v| < \varepsilon \end{array} \right\}, \\ U_\varepsilon &:= \left\{ p \in M \,\middle|\, d(p,Q) = \min_{q\in Q} d(p,q) < \varepsilon \right\}. \end{aligned} \tag{9.18}$$

They are open and, for $\varepsilon > 0$ sufficiently small, the exponential map

$$\exp : TQ_\varepsilon^\perp \to U_\varepsilon$$

is a diffeomorphism. We orient the normal bundle such that orientations match in the direct sum $T_qM = T_qQ \oplus T_qQ^\perp$ for $q \in Q$. Let $\tau_\varepsilon \in \Omega_c^n(TQ^\perp)$ be a Thom form such that

$$\mathrm{supp}(\tau_\varepsilon) \subset TQ_\varepsilon^\perp, \qquad d\tau_\varepsilon = 0, \qquad \pi_*\tau_\varepsilon = 1. \tag{9.19}$$

Such a form exists by Lemma 9.30. Now define $\tau_Q \in \Omega^n(M)$ by

$$\tau_Q := \begin{cases} (\exp^{-1})^*\tau_\varepsilon & \text{on } U_\varepsilon, \\ 0 & \text{on } M \setminus U_\varepsilon. \end{cases} \tag{9.20}$$

This form is closed, by definition. The next lemma shows that τ_Q is dual to Q as in Section 8.4.3.

Lemma 9.35. *Let $Q \subset M$ and $\tau_Q \in \Omega^n(M)$ be as above. Then*

$$\int_M \omega \wedge \tau_Q = \int_Q \omega \tag{9.21}$$

for every closed $(m-n)$-form $\omega \in \Omega^{m-n}(M)$.

Proof. Denote the inclusion of the zero section in $TQ^\perp$ by

$$\iota_Q : Q \to TQ^\perp$$

For every closed form $\omega \in \Omega^{m-n}(M)$ we compute

$$\begin{aligned}
\int_M \omega \wedge \tau_Q &= \int_{U_\varepsilon} \omega \wedge \tau_Q \\
&= \int_{TQ_\varepsilon^\perp} \exp^* \omega \wedge \tau_\varepsilon \\
&= \int_Q \iota_Q^* \exp^* \omega \\
&= \int_Q (\exp \circ \iota_Q)^* \omega \\
&= \int_Q \omega.
\end{aligned}$$

Here the third step follows from Lemma 9.28 and Remark 9.29, because the open set $TQ_\varepsilon^\perp \subset TQ^\perp$ is a star shaped open neighborhood of the zero section. The last step follows from the fact that the map

$$\exp \circ \iota_Q : Q \to M$$

is just the inclusion of Q into M. This proves the lemma. □

Although the existence of a closed $(m-n)$-form τ_Q that is dual to Q, i.e. that satisfies equation (9.21), follows already from Poincaré duality, Lemma 9.35 gives us a geometrically explicit representative of this cohomology class that is supported in an arbitrarily small neighborhood of the submanifold Q. We will now show how this explicit representative can be used to relate the cup product in cohomology to the intersection numbers of submanifolds.

Theorem 9.36. *Let $Q \subset M$ and $\tau_Q \in \Omega^{m-n}(M)$ be as in Lemma 9.35. Let P be a compact oriented smooth n-manifold without boundary and let $f : P \to M$ be a smooth map that is transverse to Q. (See Figure 9.1.) Then*

$$Q \cdot f = \int_P f^* \tau_Q. \tag{9.22}$$

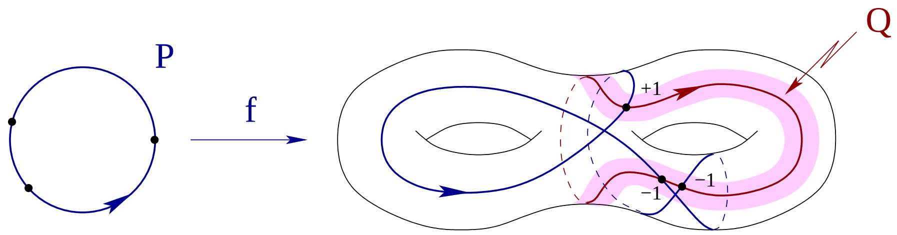


Figure 9.1: The intersection number of Q and f.

Proof. By assumption $f^{-1}(Q)$ is a finite set. We denote it by

$$f^{-1}(Q) =: \{p_1, \dots, p_k\}$$

and observe that

$$T_{f(p_i)}M = T_{f(p_i)}Q \oplus \mathrm{im}\, df(p_i), \qquad i = 1, \dots, k. \tag{9.23}$$

Since $\dim P + \dim Q = \dim M$, the derivative $df(p_i) : T_{p_i}P \to T_{f(p_i)}M$ is an injective linear map and hence its image inherits an orientation from $T_{p_i}P$. The intersection index $\iota(p_i; Q, f) \in \{\pm 1\}$ is obtained by comparing orientations in (9.23) and the intersection number of Q and f is, by definition, the sum of the intersection indices:

$$Q \cdot f = \sum_{i=1}^{k} \iota(p_i; Q, f).$$

It follows from the injectivity of $df(p_i)$ that the restriction of f to a sufficiently small neighborhood $V_i \subset P$ of p_i is an embedding. Its image is transverse to Q. Choosing $\varepsilon > 0$ sufficiently small and shrinking the V_i, if necessary, we may assume that the V_i are pairwise disjoint and that the tubular neighborhood $U_\varepsilon \subset M$ in (9.18) satisfies

$$f^{-1}(U_\varepsilon) = V_1 \cup V_2 \cup \cdots \cup V_k.$$

Since $\mathrm{supp}(\tau_Q) \subset U_\varepsilon$ we obtain $\mathrm{supp}(f^*\tau_Q) \subset f^{-1}(U_\varepsilon) = \bigcup_{i=1}^k V_i$ and hence

$$\int_P f^*\tau_Q = \sum_{i=1}^k \int_{V_i} f^*\tau_Q = \sum_{i=1}^k \int_{V_i} (\exp^{-1} \circ f)^*\tau_\varepsilon. \qquad (9.24)$$

Here the second equation uses the exponential map $\exp : TQ_\varepsilon^\perp \to U_\varepsilon$ and the Thom form $\tau_\varepsilon = \exp^* \tau_Q \in \Omega_c^n(TQ^\perp)$ with support in $TQ_\varepsilon^\perp$.

Now choose a local trivialization

$$\psi_i : TQ^\perp|_{W_i} \to W_i \times \mathbb{R}^n$$

of the normal bundle $TQ^\perp$ over a contractible neighborhood

$$W_i \subset Q$$

of $f(p_i)$ such that the open set $TQ_\varepsilon^\perp|_{W_i}$ is mapped diffeomorphically onto $W_i \times B_\varepsilon$. Here $B_\varepsilon \subset \mathbb{R}^n$ denotes the open ball of radius ε centered at zero. Let $\tau_i \in \Omega^n(W_i \times B_\varepsilon)$ be the Thom form defined by

$$\psi_i^*\tau_i = \tau_\varepsilon.$$

Then, by (9.24), we have

$$\int_P f^*\tau_Q = \sum_{i=1}^k \int_{V_i} (\exp^{-1} \circ f)^*\tau_\varepsilon = \sum_{i=1}^k \int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^*\tau_i. \qquad (9.25)$$

Consider the composition

$$f_i := \mathrm{pr}_2 \circ \psi_i \circ \exp^{-1} \circ f|_{V_i} : V_i \to B_\varepsilon.$$

If $\varepsilon > 0$ is chosen sufficiently small, this is a diffeomorphism; it is orientation preserving if $\iota(p_i; Q, f) = 1$ and is orientation reversing if $\iota(p_i; Q, f) = -1$. Since W_i is contractible there is a homotopy $h_t : V_i \to W_i$ such that

$$h_0 \equiv f(p_i), \qquad h_1 = \mathrm{pr}_1 \circ \psi_i \circ \exp^{-1} \circ f|_{V_i} : V_i \to W_i.$$

Thus

$$h_1 \times f_i = \psi_i \circ \exp^{-1} \circ f|_{V_i} : V_i \to W_i \times B_\varepsilon.$$

Moreover, the pullback of the Thom form $\tau_i \in \Omega^n(W_i \times B_\varepsilon)$ under the homotopy $h_t \times f_i$ has compact support in $[0,1] \times V_i$.

With this notation in place it follows from Corollary 7.31 and Stokes' Theorem 7.26 that

$$\begin{aligned}\int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^* \tau_i &= \int_{V_i} (h_1 \times f_i)^* \tau_i \\ &= \int_{V_i} (h_0 \times f_i)^* \tau_i \\ &= \iota(p_i; Q, f) \int_{\{f(p_i)\} \times B_\varepsilon} \tau_i \\ &= \iota(p_i; Q, f).\end{aligned}$$

Here the third equality follows from Exercise 7.25 and the last from the fact that the integral of τ_i over each slice $\{q\} \times B_\varepsilon$ is equal to one. Combining this with (9.25) we find

$$\int_P f^* \tau_Q = \sum_{i=1}^k \int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^* \tau_i = \sum_{i=1}^k \iota(p_i; Q, f) = Q \cdot f.$$

This proves the theorem. □

Proof of Theorem 8.44. By Lemma 9.35, the closed n-form $\tau_Q \in \Omega^n(M)$, constructed in (9.20) via the Thom class on the normal bundle $TQ^\perp$, is dual to Q as in Section 8.4.3. Thus Theorem 9.36 gives

$$Q \cdot f = \int_P f^* \tau_Q = \int_M \tau_Q \wedge \tau_f = (-1)^{n(m-n)} \int_Q \tau_f.$$

Here the second equation follows from the definition of the cohomology class $[\tau_f] \in H^{m-n}(M)$ dual to f in Section 8.4.3 and the last from Lemma 9.35. This proves the theorem. □

Let P and Q be compact oriented submanifolds of M without boundary and suppose that $\dim P + \dim Q = \dim M$. Then Theorem 8.44 asserts that

$$P \cdot Q = \int_M \tau_P \wedge \tau_Q.$$

By Lemma 9.35 we may choose τ_P and τ_Q with support in arbitrarily small neighborhoods of P and Q, respectively, arising from Thom forms on the normal bundles as in (9.20). If P is transverse to Q then the exterior product $\tau_P \wedge \tau_Q$ is supported near the intersection points of P and Q, and the contribution to the integral is precisely the intersection number near each intersection point. This is the geometric content of Theorem 8.44.

Example 9.37. Consider the manifold $M = \mathbb{R}^2$ and the submanifolds

$$P = \mathbb{R} \times \{0\}, \qquad Q = \{0\} \times \mathbb{R},$$

Thus P and Q are the x-axis and the y-axis, respectively, in the Euclidean plane with their standard orientations. We choose Thom forms

$$\tau_P = \rho(y)\, dy, \qquad \tau_Q = -\rho(x)\, dx,$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is a smooth compactly supported function with integral one. Then the exterior product

$$\tau_P \wedge \tau_Q = \rho(x)\rho(y) dx \wedge dy$$

is a compactly supported 2-form on $\mathbb{R}^2$ with integral one. This is also the intersection index of P and Q at the unique intersection point.

9.3 The Euler Class

9.3.1 The Euler Number

It is interesting to specialize Theorem 9.36 to the case where the manifold M is replaced by the total space of an oriented rank-n vector bundle over a compact oriented manifold M without boundary, Q is replaced by the zero section

$$Z = \{0_p \,|\, p \in M\} \subset E,$$

and $f : P \to M$ is replaced by a section

$$s : M \to E.$$

In this case the normal bundle of the submanifold Z is the vector bundle E itself (with an appropriate choice of the Riemannian metric). The dimension condition $\dim P + \dim Q = \dim M$ in intersection theory translates into the condition

$$\mathrm{rank} E = \dim M = m.$$

As before, we impose the condition that the section $s : M \to E$ is transverse to the zero section. It will be useful to take a closer look at this transversality condition.

Definition 9.38 (Vertical Differential). *Let $s : M \to E$ be a section of a vector bundle. A point $p \in M$ is called a* **zero of** s *if $s(p) = 0_p \in E_p$ is the zero element of the fiber $E_p = \pi^{-1}(p)$. The* **vertical differential of** s **at a zero** $p \in M$ *is the liner map*

$$Ds(p) : T_pM \to E_p$$

defined as follows. Let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ be a local trivialization such that $p \in U_\alpha$ and consider the vector space isomorphism

$$\psi_\alpha(p) := \mathrm{pr}_2 \circ \psi_\alpha|_{E_p} : E_p \to V$$

and the section in local coordinates

$$s_\alpha := \mathrm{pr}_2 \circ \psi_\alpha \circ s|_{U_\alpha} : U_\alpha \to V.$$

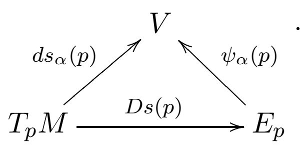
Then the vertical differential

$$Ds(p) : T_pM \to E_p$$

is defined by

$$Ds(p)v := \psi_\alpha(p)^{-1} ds_\alpha(p)v \tag{9.26}$$

for $v \in T_pM$. Thus we have the commuting diagram



The reader may check that the linear map (9.26) *is independent of the choice of α with $p \in U_\alpha$ (provided that $s(p) = 0_p$).*

Exercise 9.39. Show that there is a natural splitting of the tangent bundle of E along the zero section:

$$T_{0_p}E \cong T_pM \oplus E_p, \qquad p \in M. \tag{9.27}$$

Here the inclusion of T_pM into $T_{0_p}E$ is given by the derivative of the zero section. Show that, if $s : M \to E$ is a section and $p \in M$ is a zero of s, then the vertical differential $Ds(p) : T_pM \to E_p$ is the composition of the usual derivative $ds(p) : T_pM \to T_{0_p}E$ with the projection $T_{0_p}E \to E_p$ onto the vertical subspace in the splitting (9.27).

Exercise 9.40. Show that a section $s : M \to E$ is **transverse to the zero section** if and only if

$$s(p) = 0_p \qquad \Longrightarrow \qquad Ds(p) : T_pM \to E_p \text{ is surjective.}$$

We write $s \pitchfork 0$ to mean that s is transverse to the zero section.

Exercise 9.41. Let E be a real rank-n vector bundle over a smooth m-manifold M and let $s : M \to E$ be a smooth section of E. Assume s is transverse to the zero section. Then the **zero set**

$$s^{-1}(0) := \{p \in M \mid s(p) = 0_p\}$$

of s is a smooth submanifold of M of dimension $m - n$ and

$$T_p s^{-1}(0) = \ker\, Ds(p)$$

for every $p \in M$ with $s(p) = 0_p$.

Theorem 9.42 (**Euler Number**). *Let E be a real rank-m vector bundle over a compact oriented smooth m-manifold M without boundary and let $\tau \in \Omega_c^m(E)$ be a Thom form. Let $s : M \to E$ be a smooth section that is transverse to the zero section and define the index of a zero $p \in M$ of s by*

$$\iota(p, s) := \begin{cases} +1, & \text{if } Ds(p) : T_pM \to E_p \text{ is orientation preserving,} \\ -1, & \text{if } Ds(p) : T_pM \to E_p \text{ is orientation reversing.} \end{cases} \tag{9.28}$$

Then

$$\int_M s^*\tau = \sum_{s(p)=0_p} \iota(p, s). \tag{9.29}$$

This integral is independent of s and is called the **Euler number of** E.

Proof. The intersection index of the zero section Z with $s(M)$ at a zero p of S is $\iota(p, S)$. Hence the sum on the right in equation (9.29) is the intersection number $Z \cdot s$. With this understood the assertion follows immediately from Theorem 9.36. □

Exercise 9.43. Let $\pi : E \to M$ be as in Theorem 9.42. Define the index $\iota(p, s) \in \mathbb{Z}$ of an isolated zero of a section $s : M \to E$. Prove that equation (9.29) in Theorem 9.42 continues to hold for sections with only isolated zeros. **Hint:** See the proof of the Poincaré–Hopf Theorem.

Exercise 9.44. Prove that every vector bundle over a compact manifold has a section that is transverse to the zero section. **Hint:** Use the methods developed in Chapter 5.

By Theorem 9.42 the Euler number is the self-intersection number of the zero section in E. One can show as in Chapter 3 or Chapter 5 that the right hand side in (9.29) is independent of the choice of the section s, assuming it is transverse to the zero section, and use this to define the Euler number of E in the case $\mathrm{rank} E = \dim M$. Thus the definition of the Euler number extends to the case where E is an orientable manifold (and M is not).

Example 9.45 (Euler characteristic). Consider the special case of the tangent bundle $E = TM$ of a compact oriented manifold without boundary. A section of E is a vector field $X \in \mathrm{Vect}(M)$ and it is transverse to the zero section if and only if all its zeros are nondegenerate. Hence it follows from Theorem 9.42 that

$$\int_M X^*\tau = \sum_{X(p)=0} \iota(p, X)$$

for every nondegenerate vector field $X \in \mathrm{Vect}(M)$ and every Thom form $\tau \in \Omega_c^m(TM)$. This gives an independent proof of the part of the Poincaré–Hopf theorem which asserts that the sum of the indices of the zeros of a vector field (with only nondegenerate zeros) is a topological invariant. The Poincaré–Hopf theorem also asserts that this invariant is given by

$$\int_M X^*\tau = \chi(M) = \sum_{i=0}^{m} (-1)^i \dim H^i(M).$$

(See Theorem 8.45.) In other words, *the Euler number of the tangent bundle of M is the Euler characteristic of M.*

Exercise 9.46. Think of $\mathbb{C}\mathrm{P}^1$ as the space of all 1-dimensional complex linear subspaces $\ell \subset \mathbb{C}^2$. Fix an integer d and consider the complex line bundles $H^d \to \mathbb{C}\mathrm{P}^1$ and $H^d \to \mathbb{C}\mathrm{P}^1$ defined by

$$H^d := \frac{(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*}, \qquad [z_0 : z_1; \zeta] \equiv [\lambda z_0 : \lambda z_1; \lambda^d \zeta].$$

Here $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ denotes the multiplicative group of nonzero complex numbers. Ignoring the complex structure we can think of H^d as an oriented real rank-2 vector bundle over $\mathbb{C}\mathrm{P}^1$. Prove that the Euler nuymber of H^d is equal to d. **Hint:** Find a section of H^d that is transverse to the zero section and use (9.29). Think of $\mathbb{C}\mathrm{P}^1$ as the space of all 1-dimensional complex linear subspaces $\ell \subset \mathbb{C}^2$. Show that in this formulation $H^{-1} \to \mathbb{C}\mathrm{P}^1$ is the tautological bundle over $\mathbb{C}\mathrm{P}^1$ whose fiber over ℓ is the line ℓ itself. Show that $H \to \mathbb{C}\mathrm{P}^1$ is the bundle whose fiber over ℓ is the dual space $\mathrm{Hom}^{\mathbb{C}}(\ell, \mathbb{C})$. Show that the bundle H^d is isomorphic to H^{-d} by an isomorphism that is orientation reversing on each fiber.

9.3.2 The Euler Class

Let us now drop the condition that the rank of the bundle is equal to the dimension of the base. Instead of a characteristic number we will then obtain a characteristic deRham cohomology class. Let $\pi : E \to M$ be an oriented real rank-n bundle over a compact oriented m-manifold M without boundary. **The Euler class of** E is the deRham cohomology class

$$e(E) := [s^*\tau] = s^*\tau(E) \in H^n(M)$$

where $\tau \in \Omega_c^n(E)$ is a Thom form on E and $s : M \to E$ is a smooth section. Since any two sections of E are smoothly homotopic, it follows from Theorem 8.1 and Lemma 9.30 that the cohomology class of $s^*\tau$ is independentof the choices of s and τ. Thus the Euler class is well defined. If $m = n$ the integral of $e(E)$ over M is the Euler number, by Theorem 9.42.

Theorem 9.47 (Euler Class). *Let $\pi : E \to M$ be an oriented rank-n vector bundle over a compact oriented smooth m-manifold without boundary. Let $s : M \to E$ be a smooth section and $\tau \in \Omega_c^n(E)$ be a Thom form. If s is transverse to the zero section then, for every closed form $\omega \in \Omega^{m-n}(M)$, we have*

$$\int_M \omega \wedge s^*\tau = \int_{s^{-1}(0)} \omega. \tag{9.30}$$

Thus $[s^*\tau]$ is dual to the submanifold $s^{-1}(0)$ as in Section 8.4.3. (See below for our choice of orientation of $s^{-1}(0)$.)

Corollary 9.48. *Let $\pi : E \to M$ be an oriented real rank-n bundle over a compact oriented manifold M without boundary. If n is odd the Euler class of E vanishes.*

Proof. By Exercise 9.44 there is a section $s : M \to E$ that is transverse to the zero section. The right hand side in equation (9.30) changes sign when s is replaced by $-s$ while the left hand side remains unchanged. Hence the assertion follows. □

Proof of Theorem 9.47. Choose a Riemannian metric on M. We orient the zero set

$$Q := s^{-1}(0) = \{q \in M \,|\, s(q) = 0_q\}$$

so that orientations match in the direct sum

$$T_qM = T_qQ \oplus T_qQ^\perp$$

for every $q \in Q$. Here $T_qQ^\perp$ is oriented such that the isomorphism

$$Ds(q) : T_qQ^\perp \to E_q$$

is orientation preserving. Choose $\varepsilon > 0$ such that the map

$$\exp : TQ^\perp_\varepsilon \to U_\varepsilon$$

in (9.18) is a diffeomorphism. Since the zero set of s is contained in U_ε we can choose a neighborhood $U \subset E$ of the zero section such that

$$s^{-1}(U) \subset U_\varepsilon.$$

For example, the set $U := E \setminus s(M \setminus U_\varepsilon)$ is an open neighborhood of the zero section with this property. By Lemma 9.30 we may assume without loss of generality that our Thom form is supported in U and hence

$$\mathrm{supp}(s^*\tau) \subset s^{-1}(U) \subset U_\varepsilon.$$

The key observation is that the pullback of $s^*\tau$ under the exponential map

$$\exp : TQ^\perp_\varepsilon \to U_\varepsilon$$

defines a Thom form

$$\tau_\varepsilon := \exp^* s^*\tau \in \Omega^n_c(TQ^\perp_\varepsilon).$$

Here we extend the pullback to all of $TQ^\perp$ by setting it equal to zero on $TQ^\perp \setminus TQ^\perp_\varepsilon$. To prove that $\pi_*\tau_\varepsilon = 1$ we observe that the map

$$s \circ \exp : TQ^\perp_\varepsilon \to E$$

sends $(q, 0)$ to 0_q and agrees on the zero section up to first order with Ds. Hence we can homotop the map $s \circ \exp$ to the vector bundle isomorphism

$$Ds : TQ^\perp \to E|_Q.$$

An explicit homotopy

$$F : [0,1] \times TQ_\varepsilon^\perp \to E$$

is given by

$$F(t,q,v) := f_t(q,v) := \begin{cases} t^{-1}s(\exp_q(tv)) \in E_{\exp_q(tv)}, & \text{if } t > 0, \\ Ds(q)v, & \text{if } t = 0, \end{cases}$$

for $q \in Q = s^{-1}(0)$ and $v \in T_qM$ such that $v \perp T_qQ$ and $|v| < \varepsilon$. That F is smooth can be seen by choosing local trivializations on E. Hence F is a smooth homotopy connecting the maps

$$f_0 = Ds, \qquad f_1 = s \circ \exp.$$

Moreover, F extends smoothly to the closure of $[0,1] \times TQ_\varepsilon^\perp$ and the image of the set $[0,1] \times \partial TQ_\varepsilon^\perp$ under F does not intersect the zero section of E. Hence there is an open neighborhood $U \subset E$ of the zero section such that

$$U \cap F([0,1] \times \partial TQ_\varepsilon^\perp) = \emptyset, \qquad U \cap E|_Q \subset Ds(TQ_\varepsilon^\perp).$$

We choose the Thom form $\tau \in \Omega_c^n(E)$ with support in U. Then it follows from our choice of U that the forms $f_t^*\tau$ have uniform compact support in $TQ_\varepsilon^\perp$. Hence, for each $q \in Q$, we have

$$\begin{aligned} \int_{T_qQ_\varepsilon^\perp} \tau_\varepsilon &= \int_{T_qQ_\varepsilon^\perp} (s \circ \exp)^*\tau \\ &= \int_{T_qQ_\varepsilon^\perp} f_1^*\tau \\ &= \int_{T_qQ_\varepsilon^\perp} f_0^*\tau \\ &= 1. \end{aligned}$$

Here the last equation follows from the fact that $f_0 = Ds : TQ^\perp \to E|_Q$ is an orientation preserving vector bundle isomorphism. Thus $\tau_\varepsilon = (s \circ \exp)^*\tau$ is a Thom form on $TQ_\varepsilon^\perp$ as claimed. With this understood we deduce that

$$s^*\tau =: \tau_Q \in \Omega^n(M)$$

satisfies the conditions (9.19) and (9.20). Hence the result follows from Lemma 9.35. This proves the theorem. □

Exercise 9.49. Deduce Theorem 9.42 from Theorem 9.47 as the special case where $\mathrm{rank} E = \dim M$, so that $Q = s^{-1}(0)$ is a 0-dimensional manifold, and $\omega = 1 \in \Omega^0(M)$ is the constant function one.

Theorem 9.50. *The Euler Class has the following properties.*

(Zero) *Let $\pi : E \to M$ be an oriented real rank-n bundle over a compact oriented manifold without boundary. If E admits a nowhere vanishing section then the Euler class of E is zero.*

(Functoriality) *Let $\pi : E \to M$ be an oriented real rank-n bundle over a compact oriented manifold without boundary and let $f : M' \to M$ be a smooth map defined on another compact oriented manifold without boundary. Then the Euler class of the pullback bundle $f^*E \to M'$ is is the pullback of the Euler class:*

$$e(f^*E) = f^*e(E) \in H^n(M').$$

(Sum) *The Euler class of the Whitney sum of two oriented real vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ over a compact oriented manifold M without boundary is the cup product of the Euler classes:*

$$e(E_1 \oplus E_2) = e(E_1) \cup e(E_2).$$

Proof. If $s : M \to E$ is a nowhere vanishing section then the complement of the image of s is a neighborhood of the zero section. Hence, by Lemma 9.30, there is Thom form $\tau \in \Omega^n_c(E)$ with support contained in $E \setminus s(M)$. For this Thom form we have $s^*\tau = 0$ and this proves the *(Zero)* property.

To prove *(Functoriality)* recall that

$$f^*E = \{(p', e) \in M' \times E \,|\, f(p') = \pi(e)\}$$

and define $\widetilde{f} : f^*E \to E$ as the projection onto the second factor:

$$\widetilde{f}(p', e) := e.$$

Let $\tau \in \Omega^n_c(E)$ be a Thom form. Then $\widetilde{f}^*\tau \in \Omega^n_c(f^*E)$ is a Thom form on the pullback bundle because $\widetilde{f}$ restricts to an orientation preserving isomorphism on each fiber. Now let $s : M \to E$ be a section of E. Then there is a section $f^*s : M' \to f^*E$ defined by

$$(f^*s)(p') := (p', s(f(p'))).$$

Then $\widetilde{f} \circ (f^*s) = s \circ f : M \to f^*M$ and hence

$$(f^*s)^*\widetilde{f}^*\tau = (\widetilde{f} \circ (f^*s))^*\tau = (s \circ f)^*\tau = f^*(s^*\tau).$$

This proves *(Functoriality)* of the Euler class.

To prove the *(Sum)* property abbreviate

$$E := E_1 \oplus E_2$$

and observe that there are two obvious projections $\mathrm{pr}_i : E \to E_i$ for $i = 1, 2$. Let $n_i := \mathrm{rank}(E_i)$ and let $\tau_i \in \Omega_c^{n_i}(E_i)$ be a Thom form on E_i. Then

$$\tau := \mathrm{pr}_1^*\tau_1 \wedge \mathrm{pr}_2^*\tau_2 \in \Omega_c^{n_1+n_2}(E)$$

is a Thom form on E, by Fubini's theorem. A section $s : M \to E$ can be expressed as a direct sum $s = s_1 \oplus s_2$ of two sections $s_i : M \to E_i$. Then $\mathrm{pr}_i \circ s = s_i$ and hence

$$s^*\tau = s^*\left(\mathrm{pr}_1^*\tau_1 \wedge \mathrm{pr}_2^*\tau_2\right) = s_1^*\tau_1 \wedge s_2^*\tau_2.$$

This proves the theorem. □

9.3.3 The Product Structure on $H^*(\mathbb{C}\mathrm{P}^n)$

We examine the ring structure on the deRham cohomology of $\mathbb{C}\mathrm{P}^n$ where multiplication is the cup product with unit $1 \in H^0(M)$. We already know from Example 8.52 that the odd dimensional deRham cohomology vanishes and that $H^{2k}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{R}$ for every $k = 0, 1, \ldots, n$. Throughout we identify $\mathbb{C}\mathrm{P}^k$ with a submanifold of $\mathbb{C}\mathrm{P}^n$ when $k \le n$; thus

$$\mathbb{C}\mathrm{P}^k = \left\{[z_0 : z_1 : \cdots : z_k : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^n \,\middle|\, |z_0|^2 + \cdots + |z_k|^2 > 0\right\}.$$

In particular $\mathbb{C}\mathrm{P}^0$ is the single point $[1 : 0 : \cdots : 0]$. Let $h \in H^2(\mathbb{C}\mathrm{P}^n)$ be the class dual to the submanifold $\mathbb{C}\mathrm{P}^{n-1}$ as defined in Section 8.4.3; thus

$$\int_{\mathbb{C}\mathrm{P}^n} a \cup h = \int_{\mathbb{C}\mathrm{P}^{n-1}} a \tag{9.31}$$

for every $a \in H^{2n-2}(\mathbb{C}\mathrm{P}^n)$.

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ denote the multiplicative group of nonzero complex numbers and consider the **complex line bundle** $\pi : H \to \mathbb{C}\mathrm{P}^n$ defined as the quotient

$$H := \frac{(\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}}{\mathbb{C}^*} \to \mathbb{C}\mathrm{P}^n,$$

where the equivalence relation is given by

$$[z_0 : z_1 : \cdots : z_n; \zeta] \equiv [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n; \lambda\zeta]$$

for $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}\setminus\{0\}$, $\zeta \in \mathbb{C}$, and $\lambda \in \mathbb{C}^*$. The fibers of this bundle are one dimensional complex vector spaces; hence the term *complex line bundle.* One can also think of H as an oriented real rank-2 bundle over $\mathbb{C}\mathrm{P}^n$.

Theorem 9.51. *For $k = 0, 1, \dots, n$ define the deRham cohomology class $h^k \in H^{2k}(\mathbb{C}\mathrm{P}^n)$ as the k-fold cup product of h with itself:*

$$h^k := \underbrace{h \cup \cdots \cup h}_{k \text{ times}} \in H^{2k}(\mathbb{C}\mathrm{P}^n).$$

In particular, $h^0 = 1 \in H^0(\mathbb{C}\mathrm{P}^n)$ is the empty product and $h^1 = h$. These classes have the following properties.

(i) *h is the Euler class of the oriented real rank-2 bundle $H \to \mathbb{C}\mathrm{P}^n$.*

(ii) *The cohomology class h^k dual to the submanifold $\mathbb{C}\mathrm{P}^{n-k}$; thus, for every $a \in H^{2n-2k}(\mathbb{C}\mathrm{P}^n)$, we have*

$$\int_{\mathbb{C}\mathrm{P}^n} a \cup h^k = \int_{\mathbb{C}\mathrm{P}^{n-1}} a. \tag{9.32}$$

(iii) *For $k = 0, \dots, n$ we have*

$$\int_{\mathbb{C}\mathrm{P}^k} h^k = 1. \tag{9.33}$$

(iv) *For every compact oriented $2k$-dimensional submanifold $Q \subset \mathbb{C}\mathrm{P}^n$ without boundary we have*

$$\int_Q h^k = \mathbb{C}\mathrm{P}^{n-k} \cdot Q. \tag{9.34}$$

Proof. Geometrically one can think of $\mathbb{C}\mathrm{P}^n$ is as the set of complex one dimensional subspaces of $\mathbb{C}^{n+1}$:

$$\mathbb{C}\mathrm{P}^n = \left\{\ell \subset \mathbb{C}^{n+1} \,|\, \ell \text{ is a 1-dimensional complex subspace}\right\}.$$

The **tautological complex line bundle** over $\mathbb{C}\mathrm{P}^n$ is the bundle whose fiber over ℓ is the line ℓ itself. In this formulation H is the dual of the tautological bundle so that the fiber of H over $\ell \in \mathbb{C}\mathrm{P}^n$ is the dual space

$$H_\ell = \ell^* = \mathrm{Hom}^{\mathbb{C}}(\ell, \mathbb{C}).$$

Thus H can be identified with the set of all pairs (ℓ, ϕ) where $\ell \subset \mathbb{C}^{n+1}$ is a 1-dimensional complex subspace and $\phi : \ell \to \mathbb{C}$ is a complex linear map. (**Exercise:** Verify this.) In this second formulation every complex linear map $\Phi : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a section $s : \mathbb{C}\mathrm{P}^n \to H$ which assigns to every $\ell \in \mathbb{C}\mathrm{P}^n$ the restriction $s(\ell) := \Phi|_\ell$. An example, in our previous formulation, is the projection onto the last coordinate:

$$s([z_0 : z_1 : \cdots : z_n]) := [z_0 : z_1 : \cdots : z_n; z_n].$$

This section is transverse to the zero section and its zero set is the projectve subspace $s^{-1}(0) = \mathbb{C}\mathrm{P}^{n-1} \subset \mathbb{C}\mathrm{P}^n$ with its complex orientation. By Theorem 9.47 the Euler class $e(H) \in H^2(\mathbb{C}\mathrm{P}^n)$ is dual to the zero set of any transverse section of H. Hence it follows from from our definitions that $h := e(H)$. This proves (i).

We prove the remaining assertions. By Theorem 9.50 the restriction of h to each projective subspace $\mathbb{C}\mathrm{P}^{i+1} \subset \mathbb{C}\mathrm{P}^n$ is the Euler class of the restriction of the bundle H. Hence

$$\int_{\mathbb{C}P^{i+1}} a \cup h = \int_{\mathbb{C}\mathrm{P}^i} a$$

for every $a \in H^{2i}(\mathbb{C}\mathrm{P}^n)$. By induction, we obtain

$$\int_{\mathbb{C}\mathrm{P}^{i+k}} a \cup h^k = \int_{\mathbb{C}\mathrm{P}^i} a$$

for all $i, k \geq 0$ with $i + k \leq n$ and every $a \in H^{2i}(\mathbb{C}\mathrm{P}^n)$. With $i = n - k$ this proves (ii) and, with $i = 0$ and $a = 1 \in H^0(\mathbb{C}\mathrm{P}^n)$, this proves (iii). Now let $Q \subset \mathbb{C}\mathrm{P}^n$ is any $2k$-dimensional compact oriented submanifold without boundary and $\tau_Q \in \Omega^{2n-2k}(\mathbb{C}\mathrm{P}^n)$ be a closed form dual to Q as in Section 8.4.3. Then, by Theorem 8.44, we have

$$\int_Q h^k = \int_{\mathbb{C}\mathrm{P}^n} h^k \wedge \tau_Q = \mathbb{C}\mathrm{P}^{n-k} \cdot Q.$$

Here we have used the fact that the class h^k is dual to the submanifold $\mathbb{C}\mathrm{P}^{n-k} \subset \mathbb{C}\mathrm{P}^n$, by (ii). This proves (iv) and the theorem. □

Remark 9.52. Equation (9.32) can be viewed as a special instance of the general fact, not proved in these notes, that the the cup product of two closed forms dual to transverse submanifolds $P, Q \subset M$ is dual to the intersection $P \cap Q$ (with the appropriate careful choice of orientations). Theorem 8.44 can also be interpreted as an example of this principle.

Remark 9.53. By equation (9.34), the class $h^k \in H^{2k}(\mathbb{C}\mathrm{P}^n)$ is **integral** in the sense that the integral of h^k over every compact oriented $2k$-dimensional submanifold $Q \subset \mathbb{C}\mathrm{P}^n$ without boundary is an integer. By equation (9.33), the class h^k generates the additive subgroup of all integral classes in $H^{2k}(\mathbb{C}\mathrm{P}^n)$ (also called the **integral lattice**) in the sense that every integral cohomology class in $H^{2k}(\mathbb{C}\mathrm{P}^n)$ is an integer multiple of h^k. Here we use the fact that $H^{2k}(\mathbb{C}\mathrm{P}^n)$ is a one dimensional real vector space (see Example 8.52).

Remark 9.54. The definition of the *integral lattice* in Remark 9.53 is rather primitive but suffices for our purposes. The correct definition involves a cohomology theory over the integers such as, for example, the singular cohomology. DeRham's theorem asserts that the deRham cohomology $H^*_{\mathrm{dR}}(M)$ is isomorphic to the singular cohomology $H^*_{\mathrm{sing}}(M;\mathbb{R})$ with real coefficients. Moreover, there is an obvious homomorphism $H^*_{\mathrm{sing}}(M;\mathbb{Z}) \to H^*_{\mathrm{sing}}(M;\mathbb{R})$. The correct definition of the integral lattice $\Lambda \subset H^*_{\mathrm{dR}}(M)$ is as the subgroup (in fact subring) of those classes whose images under deRham's isomorphism in $H^*_{\mathrm{sing}}(M;\mathbb{R})$ have integral lifts, i.e. belong to the image of the homomorphism $H^*_{\mathrm{sing}}(M;\mathbb{Z}) \to H^*_{\mathrm{sing}}(M;\mathbb{R})$. The relation between these two definitions of the integral lattice is not at all obvious. It is related to the question which integral singular homology classes can be represented by submanifolds. However, in the case of $\mathbb{C}\mathrm{P}^n$ these subtleties do not play a role and we do not discuss the issue further.

Remark 9.55. Theorem 9.51 asserts that the comology class $h \in H^2(\mathbb{C}\mathrm{P}^n)$ is a multiplicative generator of $H^*(\mathbb{C}\mathrm{P}^n)$. In other words, every element $a \in H^*(\mathbb{C}\mathrm{P}^n)$ can be expressed as a sum

$$a = c_0 + c_1 h + c_2 h^2 + \cdots + c_n h^n$$

with real coefficients c_i. Think of the c_i as the coefficients of a polynomial

$$p(u) = c_0 + c_1 u + c_2 u^2 + \cdots + c_n u^n$$

in one variable, so that $a = p(h)$. Thus we have a ring isomorphism

$$\frac{\mathbb{R}[u]}{\langle u^{n+1} = 0\rangle} \longrightarrow H^*(\mathbb{C}\mathrm{P}^n) : p \mapsto p(h).$$

The integral lattice in $H^*(\mathbb{C}\mathrm{P}^n)$, as defined in Remark 9.53, is the image of the subring of polynomials with integer coefficients under this isomorphism.

We shall return to the Euler class of a real rank-2 bundle in Section 10.3.3 with an alternative definition and in Section 10.3.4 with several examples.

Chapter 10

Connections and Curvature

In this chapter we discuss connections and curvature and give an introduction to Chern–Weil theory and the Chern classes of complex vector bundles. The chapter begins in Section 10.1 by introducing the basic notions of connection and parallel transport, followed by a discussion of structure groups. In Section 10.2 we introduce the curvature of a connection, followed by a discussion of gauge transformations and flat connections. With the basic notions in place we turn to Chern–Weil theory in Section 10.3. As a first application we give another definition of the Euler class of an oriented real rank-2 bundle and discuss several examples. Our main application is the introduction of the Chern classes in Section 10.4. We list their axioms, prove their existence via Chern–Weil theory, and show that the Chern classes are uniquely determined by the axioms. Various applications of the Chern classes to geometric questions are discussed in Section 10.5. The chapter closes with a brief outlook to some deeper results in differential topology.

10.1 Connections

10.1.1 Vector Valued Differential Forms

Let $\pi : E \to M$ be a real rank-n vector bundle over a smooth m-manifold M. Fix an integer $k \geq 0$. A **differential k-form on M with values in E** is a collection of alternating k-forms

$$\omega_p : \underbrace{T_pM \times T_pM \times \cdots \times T_pM}_{k \text{ times}} \to E_p,$$

one for each $p \in M$, such that the map $M \to E : p \mapsto \omega_p(X_1(p), \ldots, X_k(p))$ is a smooth section of E for every k vector fields $X_1, \ldots, X_k \in \mathrm{Vect}(M)$.

The space of k-forms on M with values in E will be denoted by $\Omega^k(M,E)$. In particular, $\Omega^0(M,E)$ is the space of smooth sections of E. A k-form on M with values in E can also be defined as a smooth section of the vector bundle $\Lambda^k T^*M \otimes E \to M$. Thus

$$\Omega^k(M) = \Omega^0(M, \Lambda^k T^*M \otimes E).$$

Remark 10.1. The space $\Omega^k(M,E)$ of E-valued k-forms on M is a real vector space. Moreover, we can multiply an E-valued k-form on M by a smooth real valued function or by a real valued differential form on M using the pointwise exterior product. This gives a bilinear map

$$\Omega^\ell(M) \times \Omega^k(M,E) \to \Omega^{k+\ell}(M,E) : (\tau,\omega) \mapsto \tau \wedge \omega,$$

defined by the same formula as in the standard case where both forms are real valued. (See Definition 7.7.)

Remark 10.2. Let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ be a family of local trivializations of E with transitions maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$. Then every global k-form $\omega \in \Omega^k(M,\mathbb{E})$ determines a family of local vector valued k-forms

$$\omega_\alpha := \mathrm{pr}_2 \circ \psi_\alpha \circ \omega|_{U_\alpha} \in \Omega^k(U_\alpha, V). \tag{10.1}$$

These local k-forms are related by

$$\omega_\beta = g_{\beta\alpha}\omega_\alpha. \tag{10.2}$$

Conversely, every collection of local k-forms $\omega_\alpha \in \Omega^k(U_\alpha, V)$ that satisfy (10.2) determine a global k-form $\omega \in \Omega^k(M,E)$ via (10.1).

10.1.2 Connections

Let $\pi : E \to M$ be a real vector bundle over a smooth manifold. A **connection on** E is a linear map

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$$

that satisfies the Leibnitz rule

$$\nabla(fs) = f\nabla s + (df)\cdot s \tag{10.3}$$

for every $f \in \Omega^0(M)$ and every $s \in \Omega^0(M,E)$. For $p \in M$ and $v \in T_pM$ we write $\nabla_v s(p) := (\nabla s)_p(v) \in E_p$ and call this the **covariant derivative of** s **at** p **in the direction** v.

The archetypal example of a connection is the usual differential

$$d : \Omega^0(M) \to \Omega^1(M)$$

on the space of smooth real valued functions on M, thought of as sections of the trivial bundle $E = M \times \mathbb{R}$. This is a first order linear operator and the same works for vector valued functions. The next proposition shows that every connection is in a local trivialization given by a zeroth order perturbation of the operator d.

Proposition 10.3 (Connections). *Let $\pi : E \to M$ be a vector bundle over a smooth manifold with local trivializations*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$$

and transitions maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V).$$

(i) *E admits a connection.*

(ii) *For every connection ∇ on E there are 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$, called* **connection potentials**, *such that*

$$(\nabla s)_\alpha = ds_\alpha + A_\alpha s_\alpha \tag{10.4}$$

for every $s \in \Omega^0(M, E)$, where $(\nabla s)_\alpha$ and s_α are defined by (10.1). The connection potentials satisfy the condition

$$A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} \tag{10.5}$$

for all α, β. Conversely, every collection of 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ satisfying (10.5) determine a connection ∇ on E via (10.4).

(iii) *If $\nabla, \nabla' : \Omega^0(M, E) \to \Omega^1(M, E)$ are connections on E then there is a 1-form $A \in \Omega^1(M, \mathrm{End}(E))$ such that*

$$\nabla' - \nabla = A.$$

Conversely if ∇ is a connection on E then so is $\nabla + A$ for every endomorphism valued 1-form $A \in \Omega^1(M, \mathrm{End}(E))$.

Proof. The proof has six steps.

Step 1. *For every section $s \in \Omega^0(M,E)$ and every connection ∇ on E we have* $\mathrm{supp}(\nabla s) \subset \mathrm{supp}(s)$.

Let $p_0 \in M \setminus \mathrm{supp}(s)$ and choose a smooth function $f : M \to [0,1]$ such that $f = 1$ on the support of s and $f = 0$ near p_0. Then $fs = s$ and hence

$$\nabla s = \nabla(fs) = f\nabla s + (df)s.$$

The right hand side vanishes near p_0 and hence ∇s vanishes at p_0. This proves Step 1.

Step 2. *For every connection ∇ on E and every α there is a 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ satisfying* (10.4).

Fix a compact subset $K \subset U_\alpha$. We first define the restriction of A_α to K. For this we choose a basis $e_1, \dots, e_n$ of V and a smooth cutoff function $\rho : M \to [0,1]$ with support in U_α such that $\rho \equiv 1$ in a neighborhood of K. For $i = 1, \dots, n$ let $s_i : M \to E$ be the smooth section defined by

$$s_i(p) := \begin{cases} \rho(p)\psi_\alpha(p)^{-1}e_i, & \text{for } p \in U_\alpha, \\ 0, & \text{for } p \in M \setminus U_\alpha. \end{cases}$$

For $p \in K$ define the linear map $(A_\alpha)_p : T_pM \to \mathrm{End}(V)$ by

$$(A_\alpha)_p(v)\sum_{i=1}^n \lambda_i e_i := \psi_\alpha(p)\sum_{i=1}^n \lambda_i \nabla_v s_i(p)$$

for $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $v \in T_pM$. By Step 1, the linear map $(A_\alpha)_p$ is independent of the choice of ρ and hence is defined for each $p \in U_\alpha$.

If $s \in \Omega^0(M,E)$ is supported in U_α we take $K = \mathrm{supp}(s)$ and choose s_i as above. Then there are $f_i : M \to \mathbb{R}$, supported in K, such that

$$s = \sum_i f_i s_i, \qquad s_\alpha = \sum_i f_i e_i.$$

Hence, for $p \in K = \mathrm{supp}(s) \subset U_\alpha$, we have

$$\begin{aligned} (\nabla s)_\alpha(p;v) &= \psi_\alpha(p)\nabla_v s(p) = \psi_\alpha(p)\sum_i \nabla_v(f_i s_i)(p) \\ &= \psi_\alpha(p)\sum_i \Big(f_i(p)\nabla_v s_i(p) + (df_i(p)v)s_i(p)\Big) \\ &= (A_\alpha)_p(v)\sum_i f_i(p)e_i + \sum_i (df_i(p)v)e_i \\ &= (A_\alpha)_p(v)s_\alpha(p) + ds_\alpha(p)v. \end{aligned}$$

By Step 1, this continues to hold when s is not supported in U_α.

Step 3. *The* 1*-forms* $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ *in Step 2 satisfy* (10.5)*.*

By definition we have $(\nabla s)_\beta = g_{\beta\alpha}(\nabla s)_\alpha$ and hence

$$ds_\beta + A_\beta s_\beta = g_{\beta\alpha}\left(ds_\alpha + A_\alpha s_\alpha\right)$$

on $U_\alpha \cap U_\beta$. Differentiating the identity $s_\beta = g_{\beta\alpha} s_\alpha$ we obtain

$$ds_\beta = g_{\beta\alpha} ds_\alpha + \left(dg_{\beta\alpha}\right) s_\alpha$$

and hence

$$\begin{aligned} A_\beta g_{\beta\alpha} s_\alpha &= A_\beta s_\beta \\ &= g_{\beta\alpha} A_\alpha s_\alpha + g_{\beta\alpha} ds_\alpha - ds_\beta \\ &= \big(g_{\beta\alpha} A_\alpha - dg_{\beta\alpha}\big) s_\alpha \end{aligned}$$

for every (compactly supported) smooth function $s_\alpha : U_\alpha \cap U_\beta \to V$. Thus $A_\beta g_{\beta\alpha} = g_{\beta\alpha} A_\alpha - dg_{\beta\alpha}$ on $U_\alpha \cap U_\beta$ and this proves Step 3.

Step 4. *Every collection of* 1*-forms* $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ *satisfying* (10.5) *determine a connection* ∇ *on* E *via* (10.4)*.*

Reversing the argument in the proof of Step 3 we find that, for every smooth section $s \in \Omega^0(M, E)$, the local E-valued 1-form

$$T_pM \to E_p : v \mapsto \psi_\alpha(p)^{-1}\big(ds_\alpha(p)v + (A_\alpha)_p(v)s_\alpha(p)\big)$$

agrees on $U_\alpha \cap U_\beta$ with the corresponding 1-form with α replaced by β. Hence these 1-forms define a global smooth 1-form $\nabla s \in \Omega^1(M, E)$. This proves Step 4. In particular, we have now established assertion (ii).

Step 5. *We prove (iii).*

The difference of two connections ∇ and ∇' is linear over the functions, i.e. $(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)s$ for all $f \in \Omega^0(M)$ and all $s \in \Omega^0(M, E)$. We leave it to the reader to verify that such an operator $\nabla' - \nabla$ is given by multiplication with an endomorphism valued 1-form. (**Hint:** See Step 2.)

Step 6. *We prove (i).*

Choose a partition of unity $\rho_\alpha : M \to [0, 1]$ subordinate to the cover $\{U_\alpha\}_\alpha$ and define $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ by

$$A_\alpha := \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} dg_{\gamma\alpha}. \qquad (10.6)$$

These 1-forms satisfy (10.5) and hence determine a connection on E, by Step 4. This proves the proposition. □

Example 10.4. The Levi-Civita connection of a Riemannian metric is an example of a connection on the tangent bundle $E = TM$ (see [16]).

Exercise 10.5. Let $s : M \to E$ be a smooth section and $p \in M$ be a zero of s so that $s(p) = 0_p \in E_p$. Then the vertical derivative of s at p is the map

$$T_pM \to E_p : v \mapsto Ds(p)v = \nabla_v s(p)$$

for every connection ∇ on E. (See Definition 9.38.)

Just as the usual differential $d : \Omega^0(M) \to \Omega^1(M)$ extends to a family of linear operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$, so does a connection ∇ on a vector bundle E induce linear operators d^∇ on differential forms with values in E.

Proposition 10.6. *Let $\pi : E \to M$ be a vector bundle over a smooth manifold and $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ be a connection. Then there is a unique collection of operators*

$$d^\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$$

such that $d^\nabla = \nabla$ for $k = 0$ and

$$d^\nabla(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg(\tau)}\tau \wedge d^\nabla\omega \tag{10.7}$$

for every $\tau \in \Omega^*(M)$ and every $\omega \in \Omega^*(M, E)$. In the local trivializations the operator d^∇ is given by

$$\left(d^\nabla\omega\right)_\alpha = d\omega_\alpha + A_\alpha \wedge \omega_\alpha \tag{10.8}$$

for $\omega \in \Omega^k(M, E)$ and $\omega_\alpha := \mathrm{pr}_2 \circ \pi_\alpha \circ \omega|_{U_\alpha} \in \Omega^k(U_\alpha, V)$.

Proof. Define $d^\nabla\omega$ by (10.8) and use equation (10.5) to show that $d^\nabla s$ is well defined. That this operator satisfies (10.7) is obvious from the definition. That equation (10.7) determines the operator d^∇ uniquely, follows from the fact that every k-form on M with values in E can be expressed as a finite sum of products of the form $\tau_i s_i$ with $\tau_i \in \Omega^k(M)$ and $s_i \in \Omega^0(M, E)$. This proves the proposition. □

Exercise 10.7. Show that

$$(d^\nabla\omega)(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) + \omega([X, Y]) \tag{10.9}$$

for $\omega \in \Omega^1(M, E)$ and $X, Y \in \mathrm{Vect}(M)$ and

$$\begin{aligned}(d^\nabla\omega)(X, Y, Z) &= \nabla_X(\omega(Y, Z)) + \nabla_Y(\omega(Z, X)) + \nabla_Z(\omega(X, Y)) \\ &\quad - \omega(X, [Y, Z]) - \omega(Y, [Z, X]) - \omega(Z, [X, Y])\end{aligned} \tag{10.10}$$

for $\omega \in \Omega^2(M, E)$ and $X, Y, Z \in \mathrm{Vect}(M)$. **Hint:** Use (7.22) and (7.23).

10.1.3 Parallel Transport

Let ∇ be a connection on a vector bundle $\pi : E \to M$ over a smooth manifold. For every smooth path $\gamma : I \to M$ on an interval $I \subset \mathbb{R}$ the connection determines a collection of vector space isomorphisms

$$\Phi_\gamma^\nabla(t_1, t_0) : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

between the fibers of E along γ satisfying

$$\Phi_\gamma^\nabla(t_2, t_1) \circ \Phi_\gamma^\nabla(t_1, t_0) = \Phi_\gamma^\nabla(t_2, t_0), \qquad \Phi_\gamma^\nabla(t, t) = \mathrm{id} \tag{10.11}$$

for $t, t_0, t_1, t_2 \in I$. These isomorphisms are called **parallel transport of ∇ along γ** and are defined as follows.

A **section of E along γ** is a smooth map $s : I \to E$ such that $\pi \circ s = \gamma$ or, equivalently, $s(t) \in E_{\gamma(t)}$ for every $t \in I$. Thus a section of E along γ is a section of the pullback bundle $\gamma^*E \to I$ and the space of sections of E along γ will be denoted by

$$\Omega^0(I, \gamma^*E) := \{s : I \to E \,|\, \pi \circ s = \gamma\}.$$

The connection determines a linear operator

$$\nabla : \Omega^0(I, \gamma^*E) \to \Omega^0(I, \gamma^*E),$$

which is called the **covariant derivative**, as follows. In the local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ a section $s \in \Omega^0(I, \gamma^*E)$ is represented by a collection of smooth curves $s_\alpha : I_\alpha \to V$ via

$$s_\alpha(t) =: \psi_\alpha(\gamma(t))s(t) \in V, \qquad t \in I_\alpha := \gamma^{-1}(U_\alpha). \tag{10.12}$$

These curves satisfy

$$s_\beta(t) = g_{\beta\alpha}(\gamma(t))s_\alpha(t), \qquad t \in I_\alpha \cap I_\beta \tag{10.13}$$

for all α, β. Conversely, any collection of smooth curves $s_\alpha : I_\alpha \to E$ satisfying (10.13) determines a smooth section of E along γ via (10.12). The covariant derivative $\nabla s(t) \in E_{\gamma(t)}$ is defined by

$$(\nabla s)_\alpha(t) = \dot{s}_\alpha(t) + A_\alpha(\dot{\gamma}(t))s_\alpha(t), \qquad t \in I_\alpha. \tag{10.14}$$

By (10.5) we have $(\nabla s)_\beta = g_{\beta\alpha}(\gamma)(\nabla s)_\alpha$ on $I_\alpha \cap I_\beta$ and hence the vector

$$\nabla s(t) := \psi_\alpha(\gamma(t))^{-1}(\nabla s)_\alpha(t) \in E_{\gamma(t)}, \qquad t \in I_\alpha, \tag{10.15}$$

is independent of the choice of α with $\gamma(t) \in U_\alpha$.

Let us fix a smooth curve $\gamma : I \to M$ and an *initial time* $t_0 \in I$. Then it follows from the theory of linear time dependent ordinary differential equations that, for every $e_0 \in E_{\gamma(t_0)}$, there is a unique section $s \in \Omega^0(I, \gamma^*E)$ along γ satisfying the initial value problem

$$\nabla s = 0, \qquad s(t_0) = e_0. \tag{10.16}$$

This section is called the **horizontal lift of γ through e_0**.

Definition 10.8 (Parallel Transport). *The* **parallel transport of ∇ along γ from t_0 to $t \in I$** *is the linear map*

$$\Phi_\gamma^\nabla(t, t_0) : E_{\gamma(t_0)} \to E_{\gamma(t)}$$

defined by

$$\Phi_\gamma^\nabla(t, t_0)e_0 := s(t) \tag{10.17}$$

for $e_0 \in E_{\gamma(t_0)}$, where $s \in \Omega^0(I, \gamma^*E)$ is the unique horizontal lift of γ through e_0.

Exercise 10.9. Prove that parallel transport satisfies (10.11).

Exercise 10.10 (Reparametrization). If $\phi : I' \to I$ is any smooth map between intervals and $\gamma : I \to M$ is a smooth curve then

$$\Phi_{\gamma\circ\phi}^\nabla(t_1, t_0) = \Phi_\gamma^\nabla(\phi(t_1), \phi(t_0)) : E_{\gamma(\phi(t_0))} \to E_{\gamma(\phi(t_1))}$$

for all $t_0, t_1 \in I'$.

10.1.4 Structure Groups

Let $\mathrm{G} \subset \mathrm{GL}(V)$ be a Lie subgroup with Lie algebra

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G} \subset \mathrm{End}(V)$$

Let $\pi : E \to M$ be a vector bundle with structure group G, local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, and transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}$. The bundle of **endomorphisms of E** is defined by

$$\mathrm{End}(E) := \left\{ (p, \xi) \,\middle|\, \begin{array}{l} p \in M,\ \xi : E_p \to E_p \text{ is a linear map}, \\ p \in U_\alpha \implies \psi_\alpha(p) \circ \xi \circ \psi_\alpha(p)^{-1} \in \mathfrak{g} \end{array} \right\}. \tag{10.18}$$

Thus $\mathrm{End}(E)$ is a vector bundle whose fibers are isomorphic to the Lie algebra $\mathfrak{g}$. The space of sections of $\mathrm{End}(E)$ carries a Lie algebra structure, understood pointwise. Differential forms with values in $\mathrm{End}(E)$ are in local trivializations represented by differential forms on U_α with values in $\mathfrak{g}$.

Proposition 10.11. *Let $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ be a connection on E with connection potentials $A_\alpha \in \Omega^0(U_\alpha, \mathrm{End}(V))$*

(i) *The connection potentials $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ take values in $\mathfrak{g}$ if and only if parallel transport preserves the structure group, i.e. for every smooth path $\gamma : I \to M$ and all $t_0, t_1 \in I$ with $\gamma(t_0) \in U_\alpha$ and $\gamma(t_1) \in U_\beta$ we have*

$$\psi_\beta(\gamma(t_1)) \circ \Phi_\gamma^\nabla(t_1, t_0) \circ \psi_\alpha(\gamma(t_0))^{-1} \in \mathrm{G}. \tag{10.19}$$

∇ is called a G**-connection on** *E if it satisfies these equivalent conditions.*

(ii) *If ∇ is a* G*-connection and $A \in \Omega^1(M, \mathrm{End}(E))$ then $\nabla + A$ is a* G *connection. If $\nabla, \nabla' : \Omega^0(M,E) \to \Omega^1(M,E)$ are* G*-connections then*

$$\nabla' - \nabla \in \Omega^1(M, \mathrm{End}(E)).$$

(iii) *Every* G*-bundle admits a* G*-connection.*

Proof. It suffices to prove (i) for curves $\gamma : I \to U_\alpha$. If $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ then

$$\xi(t) := A_\alpha(\dot\gamma(t)) \in \mathfrak{g}$$

for every $t \in I$. Thus $\xi : I \to \mathfrak{g}$ is a smooth curve in the Lie algebra of G and hence the differential equation

$$\dot g(t) + \xi(t) g(t) = 0, \qquad g(t_0) = \mathbb{1},$$

has a unique solution $g : I \to \mathrm{G} \subset \mathrm{GL}(V)$. Now parallel transport along γ from t_0 to t is given by

$$\Phi_\gamma(t, t_0) = \psi_\alpha(\gamma(t))^{-1} \circ g(t) \circ \psi_\alpha(\gamma(t_0)) : E_{\gamma(t_0)} \to E_{\gamma(t)}$$

and hence satisfies (10.19). Reversing this argument we see that (10.19) for every smooth path $\gamma : I \to U_\alpha$ implies $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$. This proves (i). Assertion (ii) follows follows immediately from (i) and Proposition 10.3. Assertion (iii) follows from the explicit formula (10.6) in the proof of Proposition 10.3. This proves the proposition. □

Example 10.12. Let V be an oriented vector space and $\mathrm{G} = \mathrm{GL}^+(V)$ be the group of orientation preserving automorphisms of V. Vector bundles with structure group $\mathrm{GL}^+(V)$ are oriented vector bundles (see Section 9.1.5).

Example 10.13. Let V be a finite dimensional oriented real Hilbert space and $\mathrm{G} = \mathrm{SO}(V)$ be the group of orientation preserving orthogonal transformations of V. If $\pi : E \to M$ is a vector bundle with structure group $\mathrm{SO}(V)$ then the local trivializations induce orientations as well as inner products

$$E_p \times E_p \to \mathbb{R} : (e_1, e_2) \mapsto \langle e_1, e_2 \rangle_p$$

on the fibers. The inner products fit together smoothly in the sense that the map $M \to \mathbb{R} : p \mapsto \langle s_1(p), s_2(p) \rangle_p$ is smooth for every pair of smooth sections $s_1, s_2 \in \Omega^0(M, E)$. Such a family of inner products is called a **Riemannian structure** on E and a vector bundle E with a Riemannian structure is called a **Riemannian vector bundle**.

A connection ∇ on a Riemannian vector bundle $\pi : E \to M$ is called a **Riemannian connection** if it satisfies the Leibnitz rule

$$d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle \tag{10.20}$$

for all $s_1, s_2 \in \Omega^0(M, E)$. **Exercise:** Prove that every oriented Riemannian vector bundle admits a system of local trivializations whose transition maps take values in $\mathrm{SO}(V)$. Prove that Riemannian connections are the $\mathrm{SO}(V)$-connections in Proposition 10.11. In other words, a connection is Riemannian if and only if parallel transport preserves the inner product. Prove that $\mathrm{End}(E)$ is the bundle of skew-symmetric endomorphisms of E.

Example 10.14. Let V be a complex vector space and $\mathrm{G} = \mathrm{GL}_{\mathbb{C}}(V)$ be the group of complex linear automorphisms of V. If $\pi : E \to M$ is a vector bundle with structure group $\mathrm{GL}_{\mathbb{C}}(V)$ then the local trivializations induce complex structures on the fibers of the vector bundle that fit together smoothly, i.e. a vector bundle automorphism

$$J : E \to E, \qquad J^2 = -\mathbb{1}.$$

The pair (E, J) is called a **complex vector bundle**.

A connection ∇ on a complex vector bundle $\pi : E \to M$ is called a **complex connection** if it is complex linear, i.e.

$$\nabla(Js) = J\nabla s \tag{10.21}$$

for all $s \in \Omega^0(M, E)$. **Exercise:** Prove that every complex vector bundle admits a system of local trivializations whose transition maps take values in $\mathrm{GL}_{\mathbb{C}}(V)$. Prove that complex connections are the $\mathrm{GL}_{\mathbb{C}}(V)$-connections in Proposition 10.11. In other words, a connection is complex linear if and only if parallel transport is complex linear. Prove that $\mathrm{End}(E)$ is the bundle of complex linear endomorphisms of E.

Example 10.15. A **Hermitian vector space** is a complex vector space V equipped with a bilinear form

$$V \times V \to \mathbb{C} : (u, v) \mapsto \langle u, v \rangle_c$$

whose real part is an inner product and that is complex anti-linear in the first variable and complex linear in the second variable. Thus, for $u, v \in V$ and $\lambda \in \mathbb{C}$, we have

$$\langle \lambda u, v \rangle_c = \bar{\lambda} \langle u, v \rangle_c, \qquad \langle u, \lambda v \rangle_c = \lambda \langle u, v \rangle_c.$$

Such a bilinear form is called a **Hermitian form** on V. Note that the complex structure is skew-symmetric with respect to the inner product

$$\langle \cdot, \cdot \rangle := \mathrm{Re}\, \langle \cdot, \cdot \rangle_c,$$

and that any such inner product uniquely determines a Hermitian form. The group of **unitary automorphisms** of a Hermitian vector space V is

$$\mathrm{U}(V) := \{ g \in \mathrm{GL}_{\mathbb{C}}(V) \,|\, \langle gu, gv \rangle_c = \langle u, v \rangle_c \; \forall\, u, v \in V \} .$$

For $V = \mathbb{C}^n$ we use the standard notation $\mathrm{U}(n) := \mathrm{U}(\mathbb{C}^n)$.

If $\pi : E \to M$ is a vector bundle with structure group $\mathrm{U}(V)$ then the local trivializations induce Hermitian structures on the fibers of the vector bundle that fit together smoothly. Thus E is both a complex and a Riemannian vector bundle and the complex structure is skew-symmetric with respect to the Riemannian structure:

$$\langle e_1, Je_2 \rangle + \langle Je_1, e_2 \rangle = 0, \qquad e_1, e_2 \in E_p.$$

The Hermitian form on the fibers of E is then given by

$$\langle e_1, e_2 \rangle_c = \langle e_1, e_2 \rangle + \mathbf{i} \langle Je_1, e_2 \rangle, \qquad e_1, e_2 \in E_p.$$

A complex vector bundle with such a structure is called a **Hermitian vector bundle**. Every Hermitian vector bundle admits a system of local trivializations whose transition maps take values in $\mathrm{U}(V)$. Thus vector bundles with structure group $\mathrm{U}(V)$ are Hermitian vector bundles.

A connection ∇ on a Hermitian vector bundle $\pi : E \to M$ is called a **Hermitian connection** if it is complex linear and Riemannian, i.e. if it satisfies (10.20) and (10.21). Thus the Hermitian connections are the $\mathrm{U}(V)$-connections in Proposition 10.11. In other words, a connection is Hermitian if and only if parallel transport preserves the Hermitian structure. Moreover, $\mathrm{End}(E)$ is the bundle of skew-Hermitian endomorphisms of E.

Exercise 10.16. Every complex vector bundle E admits a Hermitian structure. Any two Hermitian structures on E are related by a complex linear automorphism of E. **Hint:** Let V be a complex vector space and $\mathscr{H}(V)$ be the space of Hermitian forms on V compatible with the given complex structure. Show that $\mathscr{H}(V)$ is a convex subset of a (real) vector space and that $\mathrm{GL}_{\mathbb{C}}(V)$ acts tansitively on $\mathscr{H}(V)$. Describe Hermitian structures in local trivializations.

10.1.5 Pullback Connections

Let $\pi : E \to M$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(V)$, local trivializations $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$, and transition maps

$$g_{\beta\alpha} : U_\alpha \times U_\beta \to \mathrm{G}.$$

Let ∇ be a G-connection on E with connection potentials

$$A_\alpha^\nabla \in \Omega^1(U_\alpha, \mathfrak{g}).$$

Let

$$f : M' \to M$$

be a smooth map between manifolds. We show that the pullback bundle

$$f^*E = \big\{(p;, e) \in M' \times E \,|\, f(p') = \pi(e)\big\}$$

is a G-bundle over M' and that the G connection ∇ on E induces a G-connection $f^*\nabla$ on f^*E. To see this note that the local trivializations of E induce local trivializations of the pullback bundle over $f^{-1}(U_\alpha)$ given by

$$f^*\psi_\alpha : f^*E|_{f^{-1}(U_\alpha)} \to f^{-1}(U_\alpha) \times V, \qquad (f^*\psi_\alpha)(p', e) := (p', \mathrm{pr}_2 \circ \psi_\alpha(e)).$$

Thus

$$(f^*\psi_\alpha)(p') = \psi_\alpha(f(p')) : (f^*E)_{p'} = E_{f(p')} \to V$$

for $p' \in f^{-1}(U_\alpha)$ and the resulting transition maps are given by

$$f^*g_{\beta\alpha} = g_{\beta\alpha} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to \mathrm{G}.$$

The connection potentials of the **pullback connection** $f^*\nabla$ are, by definition, the 1-forms

$$A_\alpha^{f^*\nabla} := f^*A_\alpha^\nabla \in \Omega^1(f^{-1}(U_\alpha), \mathfrak{g}).$$

Thus f^*E is a G-bundle and $f^*\nabla$ is a G-connection on f^*E.
Exercise: Show that the 1-forms $A_\alpha^{f^*\nabla}$ satisfy equation (10.5) with $g_{\beta\alpha}$ replaced by $f^*g_{\beta\alpha}$ and hence define a G-connection on f^*E.
Exercise: Show that the covariant derivative of a section along a curve is an example of a pullback connection.

10.2 Curvature

10.2.1 Definition and basic properties

In contrast to the exterior differential on differential forms, the operator d^∇ does not, in general, define a cochain complex. The failure of $d^\nabla \circ d^\nabla$ to vanish gives rise to the definition of the curvature of a connection.

Proposition 10.17. *Let $\pi : E \to M$ be a vector bundle over a smooth manifold and $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ be a connection.*
(i) *There is a unique endomorphism valued 2-form $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$, called the* **curvature of the connection** ∇, *such that*

$$d^\nabla d^\nabla s = F^\nabla s \tag{10.22}$$

for every $s \in \Omega^0(M,E)$. In local trivializations the curvature is given by

$$(F^\nabla s)_\alpha = F_\alpha s_\alpha, \qquad F_\alpha := dA_\alpha + A_\alpha \wedge A_\alpha \in \Omega^2(U_\alpha, \mathrm{End}(V)). \tag{10.23}$$

Moreover, on $U_\alpha \cap U_\beta$ we have

$$g_{\beta\alpha} F_\alpha = F_\beta g_{\beta\alpha}. \tag{10.24}$$

(ii) *For every $\omega \in \Omega^k(M,E)$ we have*

$$d^\nabla d^\nabla \omega = F^\nabla \wedge \omega. \tag{10.25}$$

(iii) *For $X, Y \in \mathrm{Vect}(M)$ and $s \in \Omega^0(M,E)$ we have*

$$F^\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s + \nabla_{[X,Y]} s. \tag{10.26}$$

(iv) *If ∇ is a G-connection then $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$. (See* (10.18)*.)*

Proof. We prove (i). Define $F_\alpha \in \Omega^2(U_\alpha, \mathrm{End}(V))$ by (10.23). Then, for every $s \in \Omega^0(M,E)$, we have

$$\begin{aligned}
(d^\nabla d^\nabla s)_\alpha &= d\big(ds_\alpha + A_\alpha s_\alpha\big) + A_\alpha \wedge \big(ds_\alpha + A_\alpha s_\alpha\big) \\
&= d(A_\alpha s_\alpha) + A_\alpha \wedge ds_\alpha + (A_\alpha \wedge A_\alpha) s_\alpha \\
&= \big(dA_\alpha + A_\alpha \wedge A_\alpha\big) s_\alpha \\
&= F_\alpha s_\alpha.
\end{aligned} \tag{10.27}$$

Hence on $U_\alpha \cap U_\beta$:

$$g_{\beta\alpha} F_\alpha s_\alpha = g_{\beta\alpha} \left(d^\nabla d^\nabla s\right)_\alpha = \left(d^\nabla d^\nabla s\right)_\beta = F_\beta s_\beta = F_\beta g_{\beta\alpha} s_\alpha.$$

This shows that the F_α satisfy equation (10.24) and therefore determine a global endomorphism valued 2-form $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$ via

$$(F^\nabla s)_\alpha := F_\alpha s_\alpha$$

for $s \in \Omega^0(M, .E)$. By (10.27) this global 2-form satisfies (10.22) and it is uniquely determined by this condition. Thus we have proved (i).

We prove (ii). Given $\tau \in \Omega^\ell(M)$ and $s \in \Omega^0(M, E)$, we have

$$\begin{aligned} d^\nabla d^\nabla(\tau s) &= d^\nabla\big((d\tau)s + (-1)^\ell \tau \wedge d^\nabla s\big) \\ &= \tau \wedge d^\nabla d^\nabla s \\ &= \tau F^\nabla s \\ &= F^\nabla \wedge (\tau s). \end{aligned}$$

Since every k-form $\omega \in \Omega^k(M, E)$ can be expressed as a finite sum of k-forms of the form τs we deduce that F^∇ satisfies (10.25) for all k. This proves (ii).

We prove (iii). Let $X, Y \in \mathrm{Vect}(M)$ and $s \in \Omega^0(M, E)$. It follows from equation 10.9 in Exercise 10.7 that

$$\begin{aligned} F^\nabla(X, Y)s &= \nabla_X\left(d^\nabla s(Y)\right) - \nabla_Y\left(d^\nabla s(X)\right) + d^\nabla s([X, Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s + \nabla_{[X,Y]} s. \end{aligned}$$

This proves (iii) and the proposition.

We prove (iv). If ∇ is a G-connection then

$$(F_\alpha)_p(u, v) = (dA_\alpha)_p(u, v) + [A_\alpha(u), A_\alpha(v)] \in \mathfrak{g}$$

for all $p \in U_\alpha$ and $u, v \in T_pM$. This proves (iv) and the proposition. □

Remark 10.18. A connection on a vector bundle $\pi : E \to M$ induces a connection on the endomorphism bundle $\mathrm{End}(E) \to M$. The corresponding operator

$$d^\nabla : \Omega^k(M, \mathrm{End}(E)) \to \Omega^{k+1}(M, \mathrm{End}(E))$$

is uniquely determined by the Leibniz rule

$$d^\nabla(\Phi s) = (d^\nabla \Phi)s + (-1)^{\deg(\Phi)} \Phi \wedge d^\nabla s$$

for $\Phi \in \Omega^k(M, \mathrm{End}(E))$ and $s \in \Omega^0(M, E)$. **Exercise:** If the operator d^∇ on $\Omega^*(M, \mathrm{End}(E))$ is defined by this formula, prove that

$$d^\nabla(\Phi \wedge \Psi) = (d^\nabla \Phi) \wedge \Psi + (-1)^{\deg(\Phi)} \Phi \wedge d^\nabla \Psi$$

for $\Phi, \Psi \in \Omega^*(M, \mathrm{End}(E))$. Deduce that the operator d^∇ on $\Omega^*(M, \mathrm{End}(E))$ arises from a connection on $\mathrm{End}(E)$.

10.2.2 The Bianchi Identity

Proposition 10.19 (**Bianchi Identity**). *Every connection* ∇ *on a vector bundle* $\pi : E \to M$ *satisfies the* **Bianchi identity**

$$d^\nabla F^\nabla = 0. \tag{10.28}$$

Proof 1. By definition of the operator

$$d^\nabla : \Omega^2(M, \mathrm{End}(E)) \to \Omega^3(M, \mathrm{End}(E))$$

we have

$$(d^\nabla F^\nabla)s = d^\nabla(F^\nabla s) - F^\nabla \wedge d^\nabla s = d^\nabla(d^\nabla d^\nabla s) - (d^\nabla d^\nabla) d^\nabla s = 0$$

for $s \in \Omega^0(M, E)$. □

Proof 2. In the local trivializations we have

$$\begin{aligned}
(d^\nabla F^\nabla s)_\alpha &= (d^\nabla F^\nabla s - F^\nabla \wedge d^\nabla s)_\alpha \\
&= d(F_\alpha s_\alpha) + A_\alpha \wedge F_\alpha s_\alpha - F_\alpha \wedge (ds_\alpha + A_\alpha s_\alpha) \\
&= \big(dF_\alpha + A_\alpha \wedge F_\alpha - F_\alpha \wedge A_\alpha\big) s_\alpha \\
&= \big(d(A_\alpha \wedge A_\alpha) + A_\alpha \wedge dA_\alpha - (dA_\alpha) \wedge A_\alpha\big) s_\alpha \\
&= 0
\end{aligned}$$

for $s \in \Omega^0(M, E)$. □

Proof 3. It follows from (10.10) that

$$\begin{aligned}
&(d^\nabla F^\nabla s)(X, Y, Z) \\
&= d^\nabla(F^\nabla s)(X, Y, Z) - (F^\nabla \wedge d^\nabla s)(X, Y, Z) \\
&= \nabla_X \big(F^\nabla(Y, Z)s\big) + \nabla_Y \big(F^\nabla(Z, X)s\big) + \nabla_Z \big(F^\nabla(X, Y)s\big) \\
&\quad - F^\nabla(X, [Y, Z])s - F^\nabla(Y, [Z, X])s - F^\nabla(Z, [X, Y])s \\
&\quad - F^\nabla(Y, Z)\nabla_X s - F^\nabla(Z, X)\nabla_Y s - F^\nabla(X, Y)\nabla_Z s \\
&= 0.
\end{aligned}$$

for $X, Y, Z \in \mathrm{Vect}(M)$ and $s \in \Omega^0(M, E)$. Here the last equation follows from (10.26) by direct calculation. □

Example 10.20. If ∇ is the Levi-Civita connection on the tangent bundle of a Riemannian manifold then (10.26) shows that $F^\nabla \in \Omega^2(M, \mathrm{End}(TM))$ is the Riemann curvature tensor and (10.28) is the second Bianchi identity.

10.2.3 Gauge Transformations

Let $\pi : E \to V$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(V)$, local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, and transition maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}.$$

A **gauge transformation** of E is a vector bundle automorphism $u : E \to E$ such that the vector space isomorphism

$$u_\alpha(p) := \psi_\alpha(p) \circ u(p) \circ \psi_\alpha(p)^{-1} : V \to V \tag{10.29}$$

is an element of G for every α and every $p \in U_\alpha$. The group

$$\mathscr{G}(E) := \left\{ u : E \to E \,|\, \psi_\alpha(p) \circ u(p) \circ \psi_\alpha(p)^{-1} \in \mathrm{G}\ \forall\, \alpha\ \forall\, p \in U_\alpha \right\},$$

of gauge transformations is called the **gauge group of** E.

In the local trivializations a gauge transformation is represented by the maps $u_\alpha : U_\alpha \to \mathrm{G}$ in (10.29). For all α and β these maps satisfy

$$g_{\beta\alpha} u_\alpha = u_\beta g_{\beta\alpha} \tag{10.30}$$

on $U_\alpha \cap U_\beta$. Conversely, every collection of smooth maps $u_\alpha : U_\alpha \to \mathrm{G}$ satisfying (10.30) determines a gauge transformation $u \in \mathscr{G}(E)$ via (10.29). The gauge group can be thought of as an infinite dimensional analogue of a Lie group with Lie algebra

$$\mathrm{Lie}(\mathscr{G}(E)) = \Omega^0(M, \mathrm{End}(E)).$$

If $\xi : M \to \mathrm{End}(E)$ is a section the pointwise exponential map gives rise to a gauge transformation $u = \exp(\xi)$. This shows that the gauge group $\mathscr{G}(E)$ is infinite dimensional (unless G is a discrete group or M is a finite set).

Let us denote the space of G-connections on E by

$$\mathscr{A}(E) := \left\{ \nabla : \Omega^0(M, E) \to \Omega^1(M, E) \,|\, \nabla \text{ is a G-connection} \right\}.$$

By Proposition 10.11 this space is nonempty and the difference of two G-connections is a 1-form on M with values in $\mathrm{End}(E)$. Thus $\mathscr{A}(E)$ is an affine space with corresponding vector space $\Omega^1(M, \mathrm{End}(E))$. The gauge group $\mathscr{G}(E)$ acts on the space of k-forms with values in E in the obvious manner by composition and it acts on the space of G-connections (contravariantly) by conjugation. We denote this action by

$$u^*\nabla = u^{-1} \circ \nabla \circ u : \Omega^0(M, E) \to \Omega^1(M, E)$$

for $\nabla \in \mathscr{A}(E)$ and $u \in \mathscr{G}(E)$. The connection potentials of $u^*\nabla$ are

$$A_\alpha^{u^*\nabla} = u_\alpha^{-1} du_\alpha + u_\alpha^{-1} A_\alpha^\nabla u_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}). \tag{10.31}$$

Lemma 10.21. *The curvature of the connection* $u^*\nabla$ *is given by*

$$F^{u^*\nabla} = u^{-1} \circ F^\nabla \circ u \in \Omega^2(M, \mathrm{End}(E)) \tag{10.32}$$

and in the local trivialisations by

$$F_\alpha^{u^*\nabla} = u_\alpha^{-1} F_\alpha^\nabla u_\alpha \in \Omega^2(U_\alpha, \mathfrak{g}).$$

The parallel transport of the connection $u^*\nabla$ *is given by*

$$\Phi_\gamma^{u^*\nabla}(t_1, t_0) = u(\gamma(t_1))^{-1} \circ \Phi_\gamma(t_1, t_0) \circ u(\gamma(t_0)) : E_{\gamma(t_0)} \to E_{\gamma(t_1)} \tag{10.33}$$

for every smooth path $\gamma : I \to M$ *and all* $t_0, t_1 \in I$.

Proof. Equation (10.32) follows directly from the definitions. To prove equation (10.33) we choose a smooth curve $\gamma : I \to U_\alpha$ and a smooth vector field $s(t) \in E_{\gamma(t)}$ along γ and abbreviate

$$\widetilde{s} := u^{-1}s, \qquad \widetilde{\nabla} := u^*\nabla, \qquad \widetilde{A}_\alpha := u_\alpha^{-1} du_\alpha + u_\alpha^{-1} A_\alpha u_\alpha.$$

In the local trivialization over U_α we have

$$s_\alpha(t) = \psi_\alpha(\gamma(t))^{-1} s(t)$$

and

$$\widetilde{s}_\alpha(t) = \psi_\alpha(\gamma(t))^{-1} u(\gamma(t)) s(t)$$

and hence

$$s_\alpha(t) = u_\alpha(\gamma(t)) \widetilde{s}_\alpha(t).$$

Differentiating this equation we obtain

$$\begin{aligned}
(\nabla s)_\alpha &= \dot{s}_\alpha + A_\alpha(\dot\gamma) s_\alpha \\
&= u_\alpha(\gamma) \frac{d}{dt} \widetilde{s}_\alpha + \big(du_\alpha(\gamma)\dot\gamma\big) \widetilde{s}_\alpha + A_\alpha(\dot\gamma) u_\alpha(\gamma) \widetilde{s}_\alpha \\
&= u_\alpha(\gamma) \left(\frac{d}{dt} \widetilde{s}_\alpha + \widetilde{A}_\alpha(\dot\gamma) \widetilde{s}_\alpha \right) \\
&= \big(u \widetilde{\nabla} \widetilde{s}\big)_\alpha.
\end{aligned}$$

Thus we have proved that

$$(u^*\nabla)(u^{-1}s) = u^{-1}(\nabla s). \tag{10.34}$$

In particular, $\nabla s \equiv 0$ if and only if $(u^*\nabla)(u^{-1}s) \equiv 0$. This proves (10.33) and the lemma □

10.2.4 Flat Connections

A connection $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ on a vector bundle $\pi : E \to M$ is called a **flat connection** if its curvature vanishes. By Proposition 10.17 a flat connection gives rise to a cochain complex

$$\Omega^0(M,E) \xrightarrow{d^\nabla} \Omega^1(M,E) \xrightarrow{d^\nabla} \Omega^2(M,E) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \Omega^m(M,E). \quad (10.35)$$

The cohomology of this complex will be denoted by

$$H^k(M,\nabla) := \frac{\ker\, d^\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)}{\mathrm{im}\, d^\nabla : \Omega^{k-1}(M,E) \to \Omega^k(M,E)}.$$

The deRham cohomology of M is the cohomology associated to the trivial connection $\nabla = d$ on the vector bundle $E = M \times \mathbb{R}$. The cohomology of the cochain complex (10.35) for a general flat connection ∇ on E is also called **deRham cohomology with twisted coefficients in** E. We shall see that a vector bundle need not admit a flat connection.

To understand flat connections geometrically, we observe that any connection ∇ on a vector bundle $\pi : E \to M$ determines a **horizontal subbundle** $H \subset TE$ of the tangent bundle of E. It is defined by

$$H_e := \left\{ \left.\frac{d}{dt}\right|_{t=0} s(t) \,\middle|\, s : \mathbb{R} \to E,\ s(0) = e,\ \nabla s \equiv 0 \right\} \quad (10.36)$$

for $e \in E$. Note that the function $s : \mathbb{R} \to E$ in this definition is a section of E along the curve $\gamma := \pi \circ s : \mathbb{R} \to M$. The image of H_e under the derivative of a local trivialization $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ with

$$p := \pi(e) \in U_\alpha$$

is the subspace

$$d\psi_\alpha(e)H_e = \{(\hat{p},\hat{v}) \in T_pM \times V \,|\, \hat{v} + (A_\alpha)_p(\hat{p})v = 0\}.$$

Here $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ is the connection potential of ∇.

Theorem 10.22. *Let ∇ be a connection on a vector bundle $\pi : E \to M$. The following are equivalent.*

(i) *The curvature of ∇ vanishes.*

(ii) *The horizontal subbundle $H \subset TE$ is involutive.*

(iii) *The parallel transport isomorphism $\Phi_\gamma^\nabla(1,0) : E_{\gamma(0)} \to E_{\gamma(1)}$ depends only on the homotopy class of $\gamma : [0,1] \to M$ with fixed endpoints.*

Proof. We prove that (i) implies (iii). Let $p_0, p_1 \in M$ and

$$[0,1] \times [0,1] \to M : (\lambda, t) \mapsto \gamma(\lambda, t) = \gamma_\lambda(t)$$

be a smooth homotopy with fixed endpoints

$$\gamma_\lambda(0) = p_0, \qquad \gamma_\lambda(1) = p_1, \qquad 0 \le \lambda \le 1.$$

Fix an element $e_0 \in E_{p_0}$ and, for $0 \le \lambda \le 1$, denote by $s_\lambda : [0,1] \to E$ the horizontal lift of γ_λ through e_0. Then it follows from the theory of ordinary differential equations that the map

$$[0,1] \times [0,1] \to E : (\lambda, t) \mapsto s(\lambda, t) := s_\lambda(t)$$

is smooth. Let $\nabla_\lambda s$ be the covariant dervative of the vector field $\lambda \mapsto s(\lambda, t)$ along the curve $\lambda \mapsto \gamma(\lambda, t)$ with t fixed and similarly with λ and t interchanged. Then

$$F^\nabla(\partial_\lambda \gamma, \partial_t \gamma)s = \nabla_\lambda \nabla_t s - \nabla_t \nabla_\lambda s \tag{10.37}$$

This is the analogue of equation (10.26) for sections along 2-parameter curves. The proof is left as an exercise for the reader. Since $\nabla_t s \equiv 0$, by defintion, and $F^\nabla \equiv 0$, by (i), we obtain

$$\nabla_t \nabla_\lambda s \equiv 0.$$

For $t = 1$ this implies that the curve $[0,1] \to E_{p_1} : \lambda \mapsto s_\lambda(1)$ is constant. Thus we have proved that (i) implies (iii).

We prove that (iii) implies (ii). Choose a Riemannian metric on M and fix an element $e_0 \in E$. Let $U_0 \subset M$ be a geodesic ball centered at $p_0 := \pi(e_0)$, whose radius is smaller than the injectivity radius r_0 of M at p_0. Then there is a unique smooth map $\xi : U_0 \to T_{p_0}M$ such that

$$\exp_{p_0}(\xi(p)) = p, \qquad |\xi(p)| < r_0$$

We define a smooth section $s : U_0 \to E$ over U_0 by

$$s(p) := \Phi_{\gamma_p}(1, 0)e_0 \in E_p, \qquad \gamma_p(t) := \exp_{p_0}(t\xi(p))$$

If $\gamma : [0,1] \to U_0$ is any smooth curve connecting p_0 to p then γ is homotopic to γ_p with fixed endpoints and hence $s(\gamma(1)) = \Phi_\gamma(1,0)e_0$. The same argument for the restriction of γ to the interval $[0,t]$ shows that

$$s(\gamma(t)) = \Phi_\gamma(t, 0)e_0, \qquad 0 \le t \le 1.$$

Differentiating this equation at $t = 1$ we obtain

$$ds(p)\dot\gamma(1) = \left.\frac{d}{dt}\right|_{t=1} s(\gamma(t)) \in H_{s(p)}.$$

This holds for every smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p_0$ and $\gamma(1) = p$. Since $\dot\gamma(1)$ can be chosen arbitrarily we obtain $\mathrm{im}\, ds(p) \subset H_{s(p)}$. Since $\dim H_{s(p)} = \dim M = \dim T_pM$ for every $p \in M$ we have

$$s(p_0) = e_0, \qquad \mathrm{im}\, ds(p) = H_{s(p)} \quad \forall\, p \in U_0.$$

Thus we have found a submanifold of E through e_0 that is tangent to H. Hence H is integrable and, by the Theorem of Frobenius, it is therefore involutive. Thus we have proved that (iii) implies (ii).

We prove that (ii) implies (i). A vector field $X \in \mathrm{Vect}(M)$ has a unique **horizontal lift** $X^\# \in \mathrm{Vect}(E)$ such that

$$d\pi \circ X^\# = X \circ \pi, \qquad X^\#(e) \in H_e \quad \forall\, e \in E.$$

We show that the Lie bracket of two such lifts is given by

$$[X^\#, Y^\#](e) = [X,Y]^\#(e) + F^\nabla\big(X(\pi(e)), Y(\pi(e))\big). \tag{10.38}$$

This equation is meaningful because $F^\nabla(X(\pi(e)), Y(\pi(e)) \in E_e \subset T_eE$. To prove (10.38) we observe that the restriction of $X^\#$ to $\pi^{-1}(U_\alpha)$ is the pullback under ψ_α of the vector field $X^\#_\alpha \in \mathrm{Vect}(U_\alpha \times V)$ given by

$$X^\#_\alpha(p,v) = (X(p), -(A_\alpha \circ X)(p)v)$$

for $p \in U_\alpha$ and $v \in V$. Hence $\mathrm{pr}_1 \circ [X^\#_\alpha, Y^\#_\alpha] = [X,Y]$ and

$$\begin{aligned}
\mathrm{pr}_2[X^\#_\alpha, Y^\#_\alpha](p,v) &= (A_\alpha \circ X)(p)(A_\alpha \circ Y)(p)v \\
&\quad -\mathcal{L}_Y(A_\alpha \circ X)(p)v \\
&\quad -(A_\alpha \circ Y)(p)(A_\alpha \circ X)(p)v \\
&\quad +\mathcal{L}_X(A_\alpha \circ Y)(p)v \\
&= [A_\alpha(X(p)), A_\alpha(Y(p))]v \\
&\quad +dA_\alpha(X(p), Y(p))v - A_\alpha([X,Y](p))v \\
&= F_\alpha(X(p), Y(p))v - A_\alpha([X,Y](p))v.
\end{aligned}$$

Here the second equation follows from (10.9) for the trivial connection on $U_\alpha \times \mathrm{End}(V)$ and the last equation follows from (10.23). This proves (10.38). It follows immediately from (10.38) that the connection ∇ is flat whenever the horizontal subbundle $H \subset TE$ is involutive. Thus we have proved that (ii) implies (i). This proves the theorem. □

Fix a vector space V and a Lie subgroup $\mathrm{G} \subset \mathrm{GL}(V)$. Every flat G-connection ∇ on a vector bundle $\pi : E \to M$ with structure group G gives rise to a group homomorphism

$$\rho^\nabla : \pi_1(M, p_0) \to \mathrm{G},$$

defined by

$$\rho^\nabla(\gamma) := \psi_\alpha(p_0) \circ \Phi_\gamma(1,0) \circ \psi_\alpha(p_0)^{-1} \in \mathrm{G} \subset \mathrm{GL}(V) \tag{10.39}$$

for every smooth loop $\gamma : [0,1] \to M$ with endpoints $\gamma(0) = \gamma(1) = p_0$. Here $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ is a local trivialization with $p_0 \in U_\alpha$. By Proposition 10.11, the right hand side of (10.39) is an element of the structure group G and, by Theorem 10.22, it depends only on the homotopy class of γ with fixed endpoints. The notation ρ^∇ is slightly misleading as the homomorphism depends on a choice of the local trivialization ψ_α. However, different choices of the local trivialization result in conjugate homomorphisms. Moreover, different choices of the base point result in conjugate representations, by equation (10.11). And Lemma 10.21 shows that the gauge group $\mathscr{G}(E)$ acts on the space $\mathscr{A}^{\mathrm{flat}}(E)$ of flat G-connections on E and that the representations ρ^∇ and $\rho^{u^*\nabla}$ are conjugate for every $\nabla \in \mathscr{A}^{\mathrm{flat}}(E)$ and every $u \in \mathscr{G}(E)$. Thus the correspondence $\nabla \mapsto \rho^\nabla$ defines a map

$$\mathscr{M}^{\mathrm{flat}}(E) := \frac{\mathscr{A}^{\mathrm{flat}}(E)}{\mathscr{G}(E)} \to \frac{\mathrm{Hom}(\pi_1(M), \mathrm{G})}{\text{conjugacy}}. \tag{10.40}$$

This map need not be bijective as different representations $\rho : \pi_1(M) \to \mathrm{G}$ may arise from flat connections on non-isomorphic G-bundles. However it extends to a bijective correspondence in the following sense.

Exercise 10.23. Prove the following assertions.

(I) For every homomorphism $\rho : \pi_1(M) \to \mathrm{G}$ there is a flat G-connection ∇ on some G-bundle $E \to M$ such that ρ^∇ is conjugate to ρ.

(II) If (E, ∇) and (E', ∇') are flat G-bundles with fibers isomorphic to V such that ρ^∇ and $\rho^{\nabla'}$ are conjugate then (E, ∇) and (E', ∇') are isomorphic. In particular, the map (10.40) is injective.

Hint: Use parallel transport to prove (II). To prove (I) choose a universal cover $\widetilde{M} \to M$ and define E as the quotient

$$E = \frac{\widetilde{M} \times V}{\pi_1(M, p_0)}.$$

Here the fundamental group acts on V throught ρ. Sections of E are ρ-equivariant maps $s : \widetilde{M} \to V$. As the additive group $\mathbb{R}$ is isomorphic to $\mathrm{GL}^+(\mathbb{R})$ via the exponential map, this gives another proof of Exercise 8.74.

10.3 Chern–Weil Theory

10.3.1 Invariant Polynomials

We assume throughout that V is a real vector space and $\mathrm{G} \subset \mathrm{GL}(V)$ is a Lie subgroup with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) \subset \mathrm{End}(V)$. An **invariant polynomial of degree** d on $\mathfrak{g}$ is a degree-d polynomial $p : \mathfrak{g} \to \mathbb{R}$ such that

$$p(g\xi g^{-1}) = p(\xi) \tag{10.41}$$

for every $\xi \in \mathfrak{g}$ and every $g \in \mathrm{G}$. The polynomial condition can be expressed as follows. Choose a basis $e_1, \ldots, e_N$ of $\mathfrak{g}$ and write the elements of $\mathfrak{g}$ as

$$\xi = \sum_{i=1}^{N} \xi^i e_i, \qquad \xi^i \in \mathbb{R}.$$

Then a polynomial of degree d on $\mathfrak{g}$ is a map of the form

$$p(\xi) = \sum_{|\nu|=d} a_\nu \xi^\nu, \qquad \xi^\nu := (\xi^1)^{\nu_1}(\xi^2)^{\nu_2}\cdots(\xi^N)^{\nu_N}, \tag{10.42}$$

where the sum runs over all multi-indices $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{N}_0^N$ satisfying

$$|\nu| := \nu_1 + \nu_2 + \cdots + \nu_N = d.$$

Definition 10.24. *Let $p : \mathfrak{g} \to \mathbb{R}$ be an invariant polynomial of degree d. Let $\pi : E \to M$ be a vector bundle with structure group* G *and local trivializations*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V.$$

Let ∇ be a G*-connection on E. We define the differential form*

$$p(F^\nabla) \in \Omega^{2d}(M)$$

as follows. Let $F_\alpha \in \Omega^2(U_\alpha, \mathfrak{g})$ be given by (10.23) *and write*

$$F_\alpha =: \sum_{i=1}^{N} \omega_\alpha^i e_i, \qquad \omega_\alpha^i \in \Omega^2(U_\alpha).$$

If p has the form (10.42) *we define*

$$p(F^\nabla)|_{U_\alpha} := \sum_{|\nu|=d} a_\nu \omega_\alpha^\nu, \qquad \omega_\alpha^\nu := (\omega_\alpha^1)^{\nu_1} \wedge (\omega_\alpha^2)^{\nu_2} \wedge \cdots \wedge (\omega_\alpha^N)^{\nu_N}.$$

It follows from (10.24) *and the invariance of p that these definitions agree on the intersection $U_\alpha \cap U_\beta$ for all α and β. The reader may verify that the differential form $p(F^\nabla) \in \Omega^{2d}(M)$ is independent of the choice of the basis of $\mathfrak{g}$ used to define it.*

10.3.2 Characteristic Classes

Theorem 10.25 (Chern–Weil). *Let $p : \mathfrak{g} \to \mathbb{R}$ be an invariant polynomial of degree d and $\pi : E \to M$ be a vector bundle with structure group* G*.*

(i) *The form $p(F^\nabla) \in \Omega^{2d}(M)$ is closed for every* G*-conection ∇ on E.*

(ii) *The deRham cohomology class of $p(F^\nabla) \in \Omega^{2d}(M)$ is independent of the choice of the* G*-conection ∇.*

(iii) *If $f : M' \to M$ is a smooth map then $p(F^{f^*\nabla}) = f^*p(F^\nabla)$.*

By Theorem 10.25 every invariant polynmial $p : \mathfrak{g} \to \mathbb{R}$ of degree d on the Lie algebra of the structure group G determines a **characteristic deRham cohomology class**

$$p(E) := [p(F^\nabla)] \in H^{2d}(M)$$

for every vector bundle $\pi : E \to M$ with structure group G. Namely, by Proposition 10.11, there is a G-connection ∇ on E and, by Theorem 10.25, the differential form $p(F^\nabla) \in \Omega^{2d}(M)$ associated to such a connection is closed and its cohomology class is independent of ∇. It follows also from Theorem 10.25 that the characteristic classes of G-bundles over different manifolds are related under pullback by smooth maps $f : M' \to M$ via

$$p(f^*E) = f^*p(E).$$

Since $p(F^\nabla) = 0$ for every flat G-connection ∇, a G-bundle with a nontrivial characteristic class does not admit a flat G-connection.

Proof of Theorem 10.25. We prove (i). The Lie bracket on $\mathfrak{g}$ determines structure constants $c_{ij}^k \in \mathbb{R}$ such that

$$[e_i, e_j] = \sum_{k=1}^{N} c_{ij}^k e_k, \qquad i, j = 1, \ldots, N.$$

It follows from the invariance of the polynomial that

$$p(\exp(t\eta)\xi \exp(-\eta)) = p(\xi)$$

for all $\xi, \eta \in \mathfrak{g}$ and all $t \in \mathbb{R}$. Differentiating this identity at $t = 0$ we obtain

$$dp(\xi)[\eta, \xi] = \left.\frac{d}{dt}\right|_{t=0} p(\exp(t\eta)\xi \exp(-\eta)) = 0.$$

For $k = 1, \dots, N$ define the polynomial $p_k : \mathfrak{g} \to \mathbb{R}$ of degree $d-1$ by

$$p_k(\xi) := dp(\xi)e_k$$

Then, for $i = 1, \dots, N$, we have

$$0 = dp(\xi)[e_i, \xi] = \sum_{j=1}^N \xi^j dp(\xi)[e_i, e_j] = \sum_{j,k=1}^N c_{ij}^k \xi^j p_k(\xi).$$

Replacing ξ by the 2-form

$$\omega_\alpha = \sum_{i=1}^N \omega_\alpha^i e_i = F_\alpha^\nabla \in \Omega^2(U_\alpha, \mathfrak{g})$$

of Definition 10.24 we obtain

$$\sum_{j,k=1}^m c_{ij}^k p_k(\omega_\alpha) \wedge \omega_\alpha^i, \qquad i = 1, \dots, N. \tag{10.43}$$

Now write the connection potentials $A_\alpha^\nabla \in \Omega^1(U_\alpha, \mathfrak{g})$ in the form

$$A_\alpha^\nabla = \sum_{i=1}^N a_\alpha^i e_i, \qquad a_\alpha^i \in \Omega^1(U_\alpha).$$

Then the Bianchi identity takes the form

$$\begin{aligned} 0 = (d^\nabla F^\nabla)_\alpha &= dF_\alpha^\nabla + [A_\alpha^\nabla \wedge F_\alpha^\nabla] \\ &= \sum_{k=1}^N (d\omega_\alpha^k) e_k + \sum_{i,j=1}^N a_\alpha^i \wedge \omega_\alpha^j [e_i, e_j] \\ &= \sum_{k=1}^N \left(d\omega_\alpha^k + \sum_{i,j=1}^N c_{ij}^k a_\alpha^i \wedge \omega_\alpha^j \right) e_k. \end{aligned}$$

Hence

$$d\omega_\alpha^k + \sum_{i,j=1}^N c_{ij}^k a_\alpha^i \wedge \omega_\alpha^j = 0, \qquad k = 1, \dots, N. \tag{10.44}$$

Combining equations (10.43) and (10.44) we obtain

$$d(p(\omega_\alpha)) = \sum_{k=1}^N p_k(\omega_\alpha) \wedge d\omega_k^\alpha = - \sum_{i,j,k=1}^N c_{ij}^k p_k(\omega_\alpha) \wedge a_\alpha^i \wedge \omega_\alpha^j = 0.$$

Here the first equation is left as an exercise for the reader, the second equation follows from (10.44), and the last equation follows from (10.43). Thus we have proved (i).

We prove (ii). Let ∇^0 and ∇^1 be two G-connections on E with connection potentials $A_\alpha^0 \in \Omega^1(U_\alpha, \mathfrak{g})$ and $A_\alpha^1 \in \Omega^1(U_\alpha, \mathfrak{g})$, respectively. Then Proposition 10.11 shows that, for $t \in \mathbb{R}$, the operator

$$\nabla^t := (1-t)\nabla^0 + t\nabla^1 : \Omega^0(M,E) \to \Omega^1(M,E)$$

is a G-connection on E with connection potentials

$$A_\alpha^t := tA_\alpha^1 + (1-t)A_\alpha^0 \in \Omega^1(U_\alpha, \mathfrak{g}).$$

Define a connection $\widetilde{\nabla}$ on the vector bundle $\widetilde{E} := E \times \mathbb{R}$ over $\widetilde{M} := M \times \mathbb{R}$ as follows. The local trivializations are given by

$$\widetilde{\psi}_\alpha : \pi^{-1}(U_\alpha) \times \mathbb{R} \to (U_\alpha \times \mathbb{R}) \times V, \qquad \widetilde{\psi}(e,t) := ((p,t), \mathrm{pr}_2 \circ \psi_\alpha(e)).$$

The connection potentials of $\widetilde{\nabla}$ in these trivializations are the 1-forms

$$\widetilde{A}_\alpha \in \Omega^1(U_\alpha \times \mathbb{R}, \mathfrak{g}), \qquad (\widetilde{A}_\alpha)_{(p,t)}(\hat{p}, \hat{t}) := (A_\alpha^t)_p(\hat{p})$$

for $p \in U_\alpha$, $\hat{p} \in T_pM$, and $t, \hat{t} \in \mathbb{R}$. Then

$$F_\alpha^{\widetilde{\nabla}} = F_\alpha^{\nabla^t} - \partial_t A_\alpha^t \wedge dt \in \Omega^2(U_\alpha \times \mathbb{R}, \mathfrak{g})$$

and hence

$$p(F^{\widetilde{\nabla}}) = \omega(t) + \tau(t) \wedge dt \in \Omega^{2d}(M \times \mathbb{R}),$$

where

$$\omega(t) := p(F^{\nabla^t}) \in \Omega^{2d}(M), \qquad t \in \mathbb{R},$$

and

$$\mathbb{R} \to \Omega^{2d-1}(M) : t \mapsto \tau(t)$$

is a smooth family of $(2d-1)$-forms on M. By (i) the $2d$-form $p(F^{\widetilde{\nabla}})$ on $\widetilde{M} = M \times \mathbb{R}$ is closed. Thus, by equation (8.12) in the proof of Lemma 8.31, we have

$$0 = d^{M\times\mathbb{R}} p(F^{\widetilde{\nabla}}) = d^M\omega(t) + \big(d^M\beta(t) + \partial_t\omega(t)\big) \wedge dt.$$

This implies $\partial_t\omega(t) = -d^M\beta(t)$ for every t and hence

$$p(F^{\nabla^1}) - p(F^{\nabla^0}) = \omega(1) - \omega(0) = \int_0^1 \partial_t om(t)\,dt = -d^M \int_0^1 \beta(t)\,dt.$$

Thus $p(F^{\nabla^1}) - p(F^{\nabla^0})$ is exact and this proves (ii).

We prove (iii). In Section 10.1.5 we have seen that the curvature of the pullback connection $f^*\nabla$ is in the local trivializations $f^*\psi_\alpha$ given by the 2-forms

$$F_\alpha^{f^*\nabla} = f^*F_\alpha^\nabla \in \Omega^1(f^{-1}(U_\alpha), \mathfrak{g}).$$

Hence it follows directly from the definitions that $p(F^{f^*\nabla}) = f^*p(F^\nabla)$. This proves (iii) and the theorem. □

10.3.3 The Euler Class of an Oriented Rank-2 Bundle

Let $\pi : E \to M$ be an oriented Riemannian real rank-2 bundle over a smooth manifold. By Example 10.13 E is a vector bundle with structure group

$$\mathrm{SO}(2) = \left\{ g = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \,\middle|\, a, c \in \mathbb{R},\, a^2 + c^2 = 1 \right\}.$$

Its Lie algebra consists of all skew-symmetric real 2×2-matrices:

$$\mathfrak{so}(2) = \left\{ \xi = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R} \right\}.$$

The linear map $e : \mathfrak{so}(2) \to \mathbb{R}$ defined by

$$e(\xi) := \frac{-\lambda}{2\pi}$$

is invariant under conjugation. (However, $e(g^{-1}\xi g) = -e(\xi)$ whenever $g \in \mathrm{O}(n)$ has determinant -1. Thus we must assume that E is oriented.) Hence there is a characteristic class

$$e(E) := [e(F^\nabla)] \in H^2(M), \tag{10.45}$$

where ∇ is Riemannian connection on E. If we change the Riemannian structure on E then there is an orientation preserving automorphism of E intertwining the two inner products. (Prove this!) Thus the characteristic class $e(E)$ is independent of the choice of the Riemannian metric. We prove below that (10.45) is the Euler class of E whenever M is a compact oriented manifold without boundary. Thus we have extended the definition of the **Euler class** of an oriented real rank-2 bundle to arbitrary base manifolds.

Theorem 10.26. *If E is an oriented real rank-2 bundle over a compact oriented manifold M without boundary then* (10.45) *is the Euler class of E.*

Proof. Choose a smooth section $s : M \to E$ that is transverse to the zero section and denote

$$Q := s^{-1}(0).$$

Choose a Riemannian metric on M and let

$$\exp : TQ_\varepsilon^\perp \to U_\varepsilon$$

be the tubular neighborhood diffeomorphism in (9.18). Multiplying s by a suitable positive function on M we may assume that

$$p \in M \setminus U_{\varepsilon/3} \qquad \Longrightarrow \qquad |s(p)| = 1.$$

Next we claim that there is a Riemannian connection ∇ on E such that

$$\nabla s = 0 \qquad \text{on} \qquad M \setminus U_{\varepsilon/2}. \tag{10.46}$$

To see this, we choose on open cover $\{U_\alpha\}$ of M such that one of the sets is $U_{\alpha_0} = M \setminus \overline{U}_{\varepsilon/3}$ and E admits a trivialization over each set U_α. In particular, we can use s to trivialize E over U_{α_0}. Next we choose a partition of unity where $\rho_{\alpha_0} = 1$ on $M \setminus U_{\varepsilon/2}$. Then the formula (10.6) in Step 6 of the proof of Proposition 10.3 defines a Riemannian connection that satisfies (10.46). It follows from (10.46) that $F^\nabla s = d^\nabla \nabla s = 0$ on $M \setminus U_{\varepsilon/2}$. Since F^∇ is a 2-form with values in the skew-symmetric endomorphisms of E we deduce that

$$F^\nabla = 0 \qquad \text{on} \qquad M \setminus U_{\varepsilon/2}. \tag{10.47}$$

The key observation is that, under this assumption, the 2-form

$$\tau_\varepsilon := \exp^* e(F^\nabla) \in \Omega_c^2(TQ_\varepsilon^\perp)$$

is a Thom form on the normal bundle of Q. With this understood we obtain from Lemma 9.35 with $\tau_Q = e(F^\nabla)$ that

$$\int_M \omega \wedge e(F^\nabla) = \int_Q \omega = \int_M \omega \wedge s^*\tau$$

for every closed form $\omega \in \Omega^{m-2}(M)$ and every Thom form $\tau \in \Omega_c^2(E)$, where the last equation follows from Theorem 9.47. By Poincaré duality in Theorem 8.38 this implies that $e(F^\nabla) - s^*\tau$ is exact, which proves the assertion. Thus it remains to prove that τ_ε is indeed a Thom form on $TQ^\perp$.

To see this, fix a point $q_0 \in Q$ and choose a positive orthonormal basis

$$u, v \in T_{q_0}Q^\perp, \qquad |u| = |v| = 1, \qquad \langle u, v\rangle = 0.$$

We define a smooth map $\gamma : \mathbb{D} \to U_\varepsilon$ on the closed unit disc $\mathbb{D} \subset \mathbb{R}^2$ by

$$\gamma(z) := \exp_{q_0}(\varepsilon(xu + yv)).$$

for $z = (x, y) \in \mathbb{D}$. (The exponential map extends to the closure of $TQ_\varepsilon^\perp$.) This is an orientation preserving embedding of $\mathbb{D}$ into a fiber of the normal bundle $\overline{TQ_\varepsilon^\perp}$ followed by the exponential map. The integral of the 2-form $e(F^\nabla)$ over γ is given by

$$\int_{\mathbb{D}} \gamma^* e(F^\nabla) = \int_{\mathbb{D}} e(F^{\gamma^*\nabla}) = 1.$$

Here the first equality follows from (iii) in Theorem 10.25 and the second equality follows from Lemma 10.27 below by choosing a positive orthonormal trivialization of the pullback bundle $\gamma^*E \to \mathbb{D}$ (for example via radial parallel transport). Hence $\pi_*\tau_\varepsilon = 1$ and this proves the theorem. □

Lemma 10.27. *Let $\mathbb{D} \subset \mathbb{R}^2$ be the closed unit disc with coordinates $z = (x, y)$ and let $s : \mathbb{D} \to \mathbb{R}^2$ and $\xi, \eta : \mathbb{D} \to \mathfrak{so}(2)$ be smooth functions. Suppose that*

$$\begin{cases} s(z) = 0, & \text{for } z = 0, \\ s(z) \neq 0, & \text{for } z \neq 0, \\ |s(z)| = 1, & \text{for } |z| \geq 1/2, \end{cases} \qquad \det(ds(0)) > 0,$$

and that the Riemannian connection

$$\nabla := d + A, \qquad A := \xi dx + \eta dy \in \Omega^1(\mathbb{D}, \mathfrak{so}(2))$$

satisfies $\nabla s = 0$ for $|z| \geq 1/2$. Then

$$\int_{\mathbb{D}} e(F^\nabla) = 1.$$

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x + \mathbf{i}y$ and think of s as a vector field on $\mathbb{D}$. For $0 \leq r < 1$ define the curve $\gamma_r : S^1 \to S^1$ by

$$\gamma_r(e^{\mathbf{i}\theta}) := s(re^{\mathbf{i}\theta}).$$

Then the index formula for vector fields shows that

$$1 = \deg(\gamma_r) = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \gamma_r(\theta)^{-1}\dot{\gamma}_r(\theta)\, d\theta, \qquad 1/2 \leq r \leq 1. \tag{10.48}$$

To see this, choose a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\gamma_r(\theta) = e^{\mathbf{i}\phi(\theta)}$ for all θ. Then $\phi(\theta + 2\pi) = \phi(\theta) + 2\pi \deg(\gamma_r)$ and this proves (10.48).

At this point it is convenien to identify $\mathfrak{so}(2)$ with the imaginary axis via the isomorphism

$$\iota : \mathfrak{so}(2) \to \mathbf{i}\mathbb{R}, \qquad \iota\left(\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}\right) := \mathbf{i}\lambda.$$

Thus $\xi \in \mathfrak{so}(2)$ acts on $\mathbb{R}^2 \cong \mathbb{C}$ by multiplication with $\iota(\xi)$ and

$$e(F^\nabla) = \frac{\mathbf{i}}{2\pi}\iota(F^\nabla) = \frac{\mathbf{i}}{2\pi}d\iota(A), \qquad \iota(A) = \iota(\xi)\,dx + \iota(\eta)\,dy.$$

The condition $\nabla s = 0$ for $|z| = 1$ takes the form

$$\partial_x s(e^{\mathbf{i}\theta}) + \iota(\xi(e^{\mathbf{i}\theta}))s(e^{\mathbf{i}\theta}) = 0, \qquad \partial_y s(e^{\mathbf{i}\theta}) + \iota(\eta(e^{\mathbf{i}\theta}))s(e^{\mathbf{i}\theta}) = 0$$

and this gives $\dot\gamma_1(\theta) = \big(\sin(\theta)\iota(\xi(e^{\mathbf{i}\theta})) - \cos(\theta)\iota(\eta(e^{\mathbf{i}\theta}))\big)\gamma_1(\theta)$. Hence

$$\begin{aligned}
\int_{\mathbb{D}} e(F^\nabla) &= \frac{\mathbf{i}}{2\pi}\int_{\mathbb{D}} d\iota(A) = \frac{\mathbf{i}}{2\pi}\int_{S^1}\big(\iota(\xi)\,dx + \iota(\eta)\,dy\big) \\
&= \frac{\mathbf{i}}{2\pi}\int_0^{2\pi}\big(\cos(\theta)\iota(\eta(e^{\mathbf{i}\theta})) - \sin(\theta)\iota(\xi(e^{\mathbf{i}\theta}))\big)\,d\theta \\
&= -\frac{\mathbf{i}}{2\pi}\int_0^{2\pi}\gamma_1(\theta)^{-1}\dot\gamma_1(\theta)\,d\theta = 1.
\end{aligned}$$

The last equation follows from (10.48) and this proves the lemma. □

Corollary 10.28. *An oriented Riemannian rank-2 vector bundle E over M admits a flat Riemannian connection if and only if its Euler class $e(E)$ vanishes in the deRham cohomology group $H^2(M)$.*

Proof. If E admits a flat Riemannian connection ∇ then $e(F^\nabla) = 0$ and hence its Euler class vanishes by Theorem 10.26. Conversely, suppose that $e(E) = 0$ and let ∇ be any Riemannian connection on E. Then $e(F^\nabla)$ is exact. Hence there is a 1-form $\alpha \in \Omega^1(M)$ such that $e(F^\nabla) = d\alpha$. Since the linear map $e : \mathfrak{so}(2) \to \mathbb{R}$ is a vector space isomorphism there is a unique 1-form $A \in \Omega^1(M, \mathrm{End}(E))$ such that $e(A) = \alpha$. Hence $\nabla - A$ is a flat Riemannian connection. This proves the corollary. □

Exercise 10.29. Let $\pi : E \to M$ be an oriented real rank-2 bundle over a connected simply connected manifold M with vanishing Euler class $e(E) = 0$ in deRham cohomology. Prove that E admits a global trivialization. **Hint:** Use the existence of a flat Riemannian connection in Corollary 10.28.

10.3.4 Two Examples

Example 10.30. Consider the vector bundle

$$E := \frac{S^2 \times \mathbb{R}^2}{\sim} \to \mathbb{R}P^2$$

where the equivalence relation on $S^2 \times \mathbb{R}^2$ is given by $(x, \zeta) \sim (-x, -\zeta)$ for $x \in S^2$ and $\zeta \in \mathbb{R}^2$. By the Borsuk–Ulam Theorem this vector bundle does not admit a nonzero section and hence has no global trivialization. It is oriented as a vector bundle (although the base manifold $\mathbb{R}\mathrm{P}^2$ is not orientable) and its Euler class vanishes in the deRham cohomology group $H^2(\mathbb{R}\mathrm{P}^2) = 0$. **Exercise:** Find a flat Riemannian connection on E.

Example 10.30 shows that the assertion of Exercise 10.29 does not extend non simply connected manifolds. The problem is that the Euler class in Chern–Weil theory is only defined with real coefficients. The definition of the Euler class can be refined with integer coefficients. This requires a cohomology theory over the integers which we do not develop here. The Euler class of an oriented rank-2 bundle is then an integral cohomology class. In particular, $H^2(\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2$ and the Euler class of the bundle in Example 10.30 is the unique nontrivial element of $H^2(\mathbb{R}P^2; \mathbb{Z})$. More generally, oriented rank-2 bundles are classified by their Euler classes in integral cohomology: two oriented rank-2 bundles over M are isomorphic if and only if they have the same Euler class in $H^2(M; \mathbb{Z})$.

Example 10.31 (Complex Line Bundles over the Torus). A complex line bundle over the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ can be described by a **cocycle**

$$\mathbb{Z}^m \to C^\infty(\mathbb{R}^m, S^1) : k \mapsto \phi_k$$

which satisfies

$$\phi_{k+\ell}(x) = \phi_\ell(x + k)\phi_k(x)$$

for $x \in \mathbb{R}^m$ and $k, \ell \in \mathbb{Z}^m$. The associated complex line bundle is

$$E_\phi := \frac{\mathbb{R}^m \times \mathbb{C}}{\mathbb{Z}^m}, \qquad [x, \zeta] \equiv [x + k, \phi_k(x)z] \quad \forall\, k \in \mathbb{Z}^m.$$

A section of E_ϕ is a smooth map $s : \mathbb{R}^m \to \mathbb{C}$ such that

$$s(x + k) = \phi_k(x)s(x)$$

for $x \in \mathbb{R}^m$ and $k \in \mathbb{Z}^m$.

A Hermitian connection on E_ϕ has the form

$$\nabla = d + A, \qquad A = \sum_{i=1}^{n} A_i(x)dx^i,$$

where the functions $A_i : \mathbb{R}^m \to \mathbf{i}\mathbb{R}$ satisfy the condition

$$A_i(x+k) - A_i(x) = -\phi_k(x)^{-1}\frac{\partial \phi_k}{\partial x^i}(x).$$

for all $x \in \mathbb{R}^m$ and all $k \in \mathbb{Z}^m$. This can be used to compute the Euler class of the bundle.

For example, any integer matrix $B \in \mathbb{Z}^{m\times m}$ determines a cocycle

$$\phi_k^B(x) = \exp(2\pi\mathbf{i}k^T Bx). \tag{10.49}$$

A Hermitian connection on E_{ϕ^B} is then given by

$$\nabla^B = d + A, \qquad A := -2\pi\mathbf{i}\sum_{i,j=1}^{m} x^i B_{ij} dx^j. \tag{10.50}$$

Its curvature is the imaginary valued 2-form

$$F^{\nabla^B} = dA = -2\pi\mathbf{i}\sum_{i<\mathbf{j}} (B_{ij} - B_{ji})\, dx^i \wedge dx^j.$$

Hence the bundle E^{ϕ^B} has the Euler class

$$e(E_{\phi^B}) = \sum_{i<j}^{m} C_{ij}[dx^i \wedge dx^j] \in H^2(\mathbb{T}^m), \qquad C := B - B^T.$$

This bundle admits a trivialization whenever B is symmetric and it admits a square root whenever B is skew-symmetric. (Prove this.) Another cocycle with the same Euler class is given by

$$\phi_k(x) = \varepsilon(k)\exp(\pi\mathbf{i}k^T Cx), \qquad \varepsilon(k+\ell) = \varepsilon(k)\varepsilon(\ell)\exp(\pi\mathbf{i}k^T C\ell),$$

with $\varepsilon(k) = \pm 1$. If $C = B - B^T$ then the numbers

$$\varepsilon(k) = \exp(\pi\mathbf{i}k^T Bk)$$

satisfy this condition.

Two cocycles ϕ and ψ are called **equivalent** if there is a function

$$u : \mathbb{R}^m \to S^1$$

such that

$$\psi_k(x) = u(x+k)^{-1}\phi_k(x)u(x)$$

for all $x \in \mathbb{R}^m$ and $k \in \mathbb{Z}^m$. We claim that every cocycle ϕ is equivalent to one of the form (10.49). To see this, we use the fact that every 2-dimensional deRham cohomology class on $\mathbb{T}^m$ with integer periods can be represented by a 2-form with constant integer coefficients (see Example 8.48). This implies that there is a skew-symmetric integer matrix $C = -C^T \in \mathbb{Z}^{m\times m}$ such that the Euler class of E_ϕ is $e(E_\phi) = \sum_{i<j} C_{ij}[dx^i \wedge dx^j]$. Now the argument in the Proof of Corollary 10.28 shows that there is Hermitian connection ∇ on E_ϕ with constant curvature

$$F^\nabla = -2\pi\mathbf{i}\sum_{i<j} C_{ij}dx^i \wedge dx^j.$$

Choose an integer matrix $B \in \mathbb{Z}^{m\times m}$ such that $C = B - B^T$ and consider the connection ∇^B in (10.50). It has the same curvature as ∇ and hence $\nabla = \nabla^B + d\xi$ for some function $\xi : \mathbb{R}^m \to \mathbf{i}\mathbb{R}$. Then $u := \exp(\xi) : \mathbb{R}^m \to S^1$ transforms ϕ^B into ϕ. **Exercise:** Fill in the details. Prove that the complex line bundles E_ϕ and E_ψ associated to equivalent cocycles are isomorphic.

10.4 Chern Classes

10.4.1 Definition and Properties

We have already used the fact that a complex Hermitian line bundle can be regarded as an oriented real rank-2 bundle. Conversely, an oriented real Riemannian rank-2 bundle has a unique complex structure compatible with the inner product and the orientation, and can therefore be considered as a **complex Hermitian line bundle**. In this setting a Hermitian connection is the same as a Riemannian connection. In the complex notation the curvature F^∇ of a Hermitian connection is an imaginary valued 2-form on M, the Bianchi identity asserts that it is closed, and the real valued closed 2-form

$$e(F^\nabla) = \frac{\mathbf{i}}{2\pi}F^\nabla \in \Omega^2(M)$$

is a representative of the Euler class. (See Lemma 10.27.) This is also the first Chern class of E, when regarded as a complex complex line bundle.

More generally, the Chern classes of complex vector bundles are characteristic classes in the even dimensional cohomology of the base manifold. They are uniquely characterized by certain axioms which we now formulate in our deRham cohomology setting. We will see that, in order to compute the Chern classes of specific vector bundles, it suffices in many cases to know that they exist and which axioms they satisfy, without knowing how they are constructed. Just as in the case of the Euler class, the definition of the Chern classes can be extended to cohomology theories with integer coefficients, but this goes beyond the scope of the present manuscript.

Theorem 10.32 (**Chern Class**). *There is a unique functor, called the* **Chern class**, *which assigns to every complex rank-n bundle* $\pi : E \to M$ *over a compact manifold a deRham cohomology class*

$$c(E) = c_0(E) + c_1(E) + \cdots + c_n(E) \in H^*(M)$$

with

$$c_i(E) \in H^{2i}(M), \qquad c_0(E) = 1,$$

and satisfies the following axioms.

(Naturality) *Isomorphic vector bundles over* M *have the same Chern class.*

(Zero) *The Chern class of the trivial bundle* $E = M \times \mathbb{C}^n$ *is* $c(E) = 1$.

(Functoriality) *The Chern class of the pullback of a complex vector bundle* $\pi : E \to M$ *under a smooth map is the pullback of the Chern class of* E*:*

$$c(f^*E) = f^*c(E).$$

(Sum) *The Chern class of the Whitney sum* $E_1 \oplus E_2$ *of two complex vector bundles over* M *is the cup product of the Chern classes:*

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

(Euler Class) *The top Chern class of a complex rank-n bundle* $\pi : E \to M$ *over a compact oriented manifold* M *without boundary is the Euler class*

$$c_n(E) = e(E).$$

It follows from the *(Euler Class)* axiom that the anti-tautological line bundle $H \to \mathbb{C}\mathrm{P}^n$ with fiber $H_\ell = \ell^*$ over $\ell \in \mathbb{C}\mathrm{P}^n$ has first Chern class

$$c_1(H) = h \in H^2(\mathbb{C}\mathrm{P}^n) \tag{10.51}$$

where h is the positive integral generator of $H^2(CP^n)$ whose integral over the submanifold $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^n$ with its complex orientation is equal to one. (See Theorem 9.51.) In fact, the proof of Theorem 10.32 shows that the *Euler Class*) axiom can be replaced by the (*Normalization*) axiom (10.51).

10.4.2 Construction of the Chern Classes

We now give an explicit construction of the Chern classes via Chern–Weil theory which works equally well for arbitrary base manifolds M, compact or not. We observe that every complex vector bundle E admits a Hermitian structure and that any two Hermitian structures on E are related by a complex automorphism of E (see Example 10.15 and Exercise 10.16). A Hermitian vector bundle of complex rank n is a vector bundle with structure group

$$\mathrm{G} = \mathrm{U}(n) = \left\{ g \in \mathbb{C}^{n\times n} \,|\, g^*g = \mathbb{1} \right\}.$$

Here $g^* := \bar{g}^T$ denotes the conjugate transpose of $g \in \mathbb{C}^{n\times n}$. The Lie algebra of $\mathrm{U}(n)$ is the real vector space of skew-Hermitian complex $n \times n$-matrices

$$\mathfrak{g} = \mathfrak{u}(n) = \left\{ \xi \in \mathbb{C}^{n\times n} \,|\, \xi^* + \xi = \mathbb{1} \right\}.$$

The eigenvalues of a matrix $\xi \in \mathfrak{u}(n)$ are imaginary and those of the matrix $\mathbf{i}\xi/2\pi$ are real. The kth **Chern polynomial**

$$c_k : \mathfrak{u}(n) \to \mathbb{R}$$

is defined as the kth symmetric function in the eigenvalues of $\mathbf{i}\xi/2\pi$. Thus

$$c_k(\xi) := \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k}$$

where the real numbers $x_1, \ldots, x_n$ denote the eigenvalues of $\mathbf{i}\xi/2\pi$ with repetitions according to multiplicity. In particular, we have

$$\begin{aligned} c_0(\xi) &= 1, \\ c_1(\xi) &= \sum_i x_i = \operatorname{trace}\left(\frac{\mathbf{i}\xi}{2\pi}\right), \\ c_2(\xi) &= \sum_{i<j} x_i x_j, \\ c_n(\xi) &= x_1x_2\cdots x_n = \det\left(\frac{\mathbf{i}\xi}{2\pi}\right). \end{aligned}$$

Thus $c_k : \mathfrak{u}(n) \to \mathbb{R}$ is an invariant polynomial of degree k and we define the **kth Chern class** of a rank-n Hermitian vector bundle $\pi : E \to M$ by

$$c_k(E) := [c_k(F^\nabla)] \in H^{2k}(M), \tag{10.52}$$

where ∇ is a Hermitian connection on E. By Theorem 10.25 this cohomology class is independent of the choice of the Hermitian connection ∇. We will now prove that these classes satisfy the axioms of Theorem 10.32.

10.4.3 Proof of Existence and Uniqueness

We begin with a technical lemma which will be needed later in the proof. It is the only place where the compactness assumption on the base enters the proof of Theorem 10.32

Lemma 10.33. *Every complex vector bundle over a compact manifold M admits an embedding into the trivial bundle $M \times \mathbb{C}^N$ for some $N \in \mathbb{N}$.*

Proof. Let $\pi : E \to M$ be a complex rank-n bundle over a compact manifold. Choose a system of local trivializations

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n, \qquad i = 1, \dots, \ell,$$

such that the U_i cover M, and a partition of unity $\rho_i : M \to [0,1]$ subordinate to this cover. Define the map $\iota : E \to M \times \mathbb{C}^{\ell n}$ by

$$\iota(e) := \Big(\pi(e), \rho_1(\pi(e))\mathrm{pr}_2(\psi_1(e)), \dots, \rho_n(\pi(e))\mathrm{pr}_2(\psi_n(e))\Big)$$

This map is a smooth injective immersion (verify this), restricts to a linear embedding into $\{p\} \times \mathbb{C}^{\ell n}$ on each fiber E_p, and it is proper (verify this as well). This proves the lemma. □

Proof of Theorem 10.32. The cohomology classes (10.52) are well defined invariants of complex vector bundles, because every complex vector bundle admits a Hermitian structure and any two Hermitian structures on a complex vector bundle are isomorphic (see Exercise 10.16). That these classes satisfy the *(Naturality)* and *(Zero)* axioms follow directly from the definitions and that they satisfy the *(Functoriality)* axiom follows immediately from Theorem 10.25. To prove the *(Sum)* axiom we observe that the Chern polynomials are the coefficients of the characteristic polynomial

$$p_t(\xi) := \det\left(\mathbb{1} + t\frac{\mathbf{i}\xi}{2\pi}\right) = \sum_{k=0}^{n} c_k(\xi)t^k.$$

In particular, for $t = 1$, we have

$$c(\xi) = \sum_{k=0}^{n} c_k(\xi) = \prod_{i=1}^{n}(1 + x_i) = \det\left(\mathbb{1} + \frac{\mathbf{i}\xi}{2\pi}\right)$$

and hence

$$c(\xi \oplus \eta) = c(\xi)c(\eta)$$

for the direct sum of two skew-Hermitian matrices. This implies

$$c(F^{\nabla_1\oplus\nabla_2}) = c(F^{\nabla_1} \oplus F^{\nabla_2}) = c(F^{\nabla_1}) \wedge c(F^{\nabla_2})$$

for the direct sum of two Hermitian connections on two Hermitian vector bundles over M and this proves the *(Sum)* axiom.

It remains to prove the *(Euler Class)* axiom. By Theorem 10.26 the first Chern class of a complex line bundle is equal to the Euler class in $H^2(M)$. With this understood, it follows from the (*Sum*) axiom for the Euler class (Theorem 9.50) and for the Chern class (already established) that the *(Euler Class*) axiom holds for Whitney sums of complex line bundles.

An example is the partial flag manifold

$$\mathcal{F}(n,N) := \left\{ (\Lambda_i)_{i=0}^n \;\middle|\; \begin{array}{l} \Lambda_i \text{ is a complex subspace of } \mathbb{C}^N, \\ \dim_{\mathbb{C}} \Lambda_i = i,\ \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n \end{array} \right\}.$$

There is a complex rank-n bundle $E(n,N) \to \mathcal{F}(n,N)$ whose fiber over the flag $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n$ is the subspace Λ_n. It is a direct sum of the complex line bundles $L_i \to \mathcal{F}(n,N)$, $i = 1, \dots, n$, whose fiber over the same flag is the intersection $\Lambda_i \cap \Lambda_{i-1}^\perp$. Hence it follows from what we have already proved that the top Chern class of the bundle $E(n,N) \to \mathcal{F}(n,N)$ agrees with its Euler class:

$$c_n(E(n,N)) = e(E(n,N)) \in H^{2n}(\mathcal{F}(n,N)).$$

Now consider the Grassmannian

$$\mathrm{G}_n(\mathbb{C}^N) := \left\{ \Lambda \subset \mathbb{C}^N \,|\, \Lambda \text{ is an } n\text{-dimensional complex subspace} \right\}$$

of complex n-planes in $\mathbb{C}^N$. It carries a tautological bundle

$$E_n(\mathbb{C}^N) \to \mathrm{G}_n(\mathbb{C}^N)$$

whose fiber over an n-dimensional complex subspace $\Lambda \subset \mathbb{C}^N$ is the subspace itself. There is an obvious map

$$\pi : \mathcal{F}(n,N) \to \mathrm{G}_n(\mathbb{C}^N)$$

which sends a partial flag $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n$ in $\mathbb{C}^N$ with $\dim_{\mathbb{C}} \Lambda_i = i$ to the subspace Λ_n. We have

$$\pi^* E_n(\mathbb{C}^n) = E(n,N)$$

and hence, by *(Functoriality)*,

$$\pi^* c_n(E_n(\mathbb{C}^N)) = c_n(E(n,N)) = e(E(n,N)) = \pi^* e(E_n(\mathbb{C}^N)).$$

At this point we use (without proof) the fact that the map

$$\pi^* : H^*(\mathrm{G}_n(\mathbb{C}^N)) \to H^*(\mathcal{F}(n,N)) \tag{10.53}$$

is injective. This implies

$$c_n(E_n(\mathbb{C}^N)) = e(E_n(\mathbb{C}^N)) \in H^{2n}(\mathrm{G}_n(\mathbb{C}^N)) \tag{10.54}$$

for every pair of integers $N \geq n \geq 0$.

By Lemma 10.33 below, a complex line bundle $\pi : E \to M$ over a compact manifold can be embedded into the trivial bundle $M \times \mathbb{C}^N$ for a suitable integer $N \in \mathbb{N}$. Such an embedding can be expressed as a smooth map

$$f : M \to \mathrm{G}_n(\mathbb{C}^N)$$

into the Grassmannian of complex n-planes in $\mathbb{C}^N$ such that E is isomorphic to the pullback of the tautological bundle $E_n(\mathbb{C}^N) \to \mathrm{G}_n(\mathbb{C}^N)$. Hence it follows from (10.54) and *(Functoriality)* that

$$c_n(E) = f^* c_n(E_n(\mathbb{C}^N)) = f^* e(E_n(\mathbb{C}^N)) = e(E).$$

This proves the existence of Chern classes satisfying the five axioms.

To prove uniqueness, we first observe that the Chern classes of complex line bundles over compact oriented manifolds without boundary are determined by the *(Euler Class)* axiom. Second, the Chern classes of the bundle $E(n,N)$ are determined by those of line bundles via the *(Naturality)* and *(Sum)* axioms, as it is isomorphic to a direct sum of complex line bundles. Third, the Chern classes of the bundle $E_n(\mathbb{C}^N)$ are determined by those of $E(n,N)$ via *(Functoriality)*, because the homomorphism (10.53) is injective. Fourth, the Chern classes of any complex vector bundle E over a compact manifold M are determined by those of $E_n(\mathbb{C}^N)$ via *(Naturality)* and *(Functoriality)*, as there is a map

$$f : M \to \mathrm{G}_n(\mathbb{C}^N)$$

for some N such that E is isomorphic to the pullback bundle $f^* E_n(\mathbb{C}^N)$:

$$E \cong f^* E_n(\mathbb{C}^N).$$

This proves the theorem. □

We remark that the map

$$\pi : \mathcal{F}(n,N) \to \mathrm{G}_n(\mathbb{C}^N)$$

is a fibration with fibers diffeomorphic to the flag manifold $\mathcal{F}(n,n)$. One can use the spectral sequence of this fibration to prove that the map (10.53) is injective. This can be viewed as an extension of the Künneth formula, but it goes beyond the scope of the present manuscript. For details see Bott and Tu [2].

We also remark that Theorem 10.32 continues to hold for noncompact base manifolds M. The only place where we have used compactness of M is in Lemma 10.33, which in turn was used for proving uniqueness. If we replace the Grassmannian with the classifying space of the unitary group $\mathrm{U}(n)$ (which can be represented as the direct limit of the Grassmanians $\mathrm{G}_n(\mathbb{C}^N)$ as N tends to ∞), then complex rank-n bundles over noncompact manifolds M can be represented as pullbacks of the tautological bundle under maps to this classifying space or, equivalently, be embedded into the product of M with an infinite dimensional complex vector space. This can be used to extend Theorem 10.32 to complex vector bundles over noncompact base manifolds or, in fact, over arbitrary topological spaces.

Exercise 10.34 (Euler Number). Let $\pi : E \to M$ be a complex rank-n bundle over compact oriented $2n$-manifold without boundary. Show directly that the top Chern number

$$\int_M c_n(E) = \int_M \det\left(\frac{\mathbf{i}}{2\pi}F^\nabla\right) = \sum_{s(p)=0_p} \iota(p,s)$$

is the Euler number of E, without using the *(Euler Class)* axiom. **Hint:** Assume s is transverse to the zero section and let p_i be the zeros of s. Show that s can be chosen with norm one outside of a disjoint collection of neighborhoods U_i of the p_i and that the connection can be chosen such that $\nabla s = 0$ on the complement of the U_i. Show that

$$\det(\mathbf{i}F^\nabla/2\pi) = 0 \qquad \text{on} \quad M \setminus \bigcup_i U_i.$$

Now use the argument in the proof Lemma 10.27 to show that

$$\int_{U_i} \det\left(\frac{\mathbf{i}}{2\pi}F^\nabla\right) = \iota(p_i,s)$$

for each i.

Exercise 10.35 (First Pontryagin Class). Let $\pi : E \to M$ be a real vector bundle and consider the tensor product $E \otimes_{\mathbb{R}} \mathbb{C}$. This is a complex vector bundle and **Pontryagin classes** of E are defined as the even Chern classes of $E \otimes_{\mathbb{R}} \mathbb{C}$:

$$p_i(E) := (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X).$$

Show that the odd Chern classes of $E \otimes_{\mathbb{R}} \mathbb{C}$ vanish. Show that

$$p_1(E) = c_1(E)^2 - 2c_2(E)$$

whenever E is itself a complex vector bundle. If E is a Hermitian vector bundle and ∇ is a Hermitian connection on E show that the first Pontryagin class can be represented by the real valued closed 4-form $\frac{1}{4\pi}\mathrm{trace}(F^\nabla \wedge F^\nabla)$:

$$p_1(E) = \frac{1}{4\pi}\left[\mathrm{trace}\left(F^\nabla \wedge F^\nabla\right)\right] \in H^4(M). \tag{10.55}$$

Hint: The endomorphism valued 4-form $F^\nabla \wedge F^\nabla \in \Omega^4(M, \mathrm{End}(E))$ is defined like the exterior product of scalar 2-forms, with the product of real numbers replaced by the composition of endomorphisms. Express the 4-form (10.55) in the form $p_1(F^\nabla)$ for a suitable invariant degree-2 polynomial $p_1 : \mathfrak{u}(2) \to \mathbb{R}$.

10.4.4 Tensor Products of Complex Line Bundles

Let

$$\pi_1 : E_1 \to M, \qquad \pi_2 : E_2 \to M$$

be complex line bundles and consider the tensor product

$$E := E_1 \otimes E_2 := \left\{ (p, e_1 \otimes e_2) \,\middle|\, \begin{array}{l} p \in M,\ e_1 \in E_1,\ e_2 \in E_2, \\ \pi_1(e_1) = \pi_2(e_2) = p \end{array} \right\}.$$

This is again a complex line bundle over M and its first Chern class is the sum of the first Chern classes of E_1 and E_2:

$$c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2). \tag{10.56}$$

(Here we use the formula (10.52) as the definition of the first Chern class in the case of a noncompact base manifold.) To see this, we choose Hermitian structures on E_1 and E_2 and Hermitian local trivializations over an open cover $\{U_\alpha\}_\alpha$ of M with transition maps $g_{i,\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{U}(1) = S^1$. These give rise, in an obvious manner, to a Hermitian structure on the tensor product $E = E_1 \otimes E_2$ and to local trivializations of E with transition maps

$$g_{\beta\alpha} = g_{1,\beta\alpha} \cdot g_{2,\beta\alpha} : U_\alpha \cap U_\beta \to S^1.$$

For $i = 1, 2$ choose a Hermitian connection ∇_i on E_i with connection potentials $A_{i,\alpha} \in \Omega^1(U_\alpha, \mathbf{i}\mathbb{R})$. They determine a connection ∇ on E via the Leibnitz rule

$$\nabla(s_1 \otimes s_2) := (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2)$$

for $s_1 \in \Omega^0(M, E_1)$ and $s_2 \in \Omega^0(M, E_2)$. The connection potentials of ∇ are

$$A_\alpha = A_{1,\alpha} + A_{2,\alpha} \in \Omega^1(U_\alpha, \mathbf{i}\mathbb{R}).$$

Hence the curvature of F^∇ is given by

$$F^\nabla = F^{\nabla_1} + F^{\nabla_2} \in \Omega^2(M, \mathbf{i}\mathbb{R}).$$

In fact, the restriction of F^∇ to U_α is just the differential of A_α. Since $c_1(E)$ is the cohomology class of the real valued closed 2-form $\frac{\mathbf{i}}{2\pi} F^\nabla \in \Omega^2(M)$, this implies equation (10.56).

Example 10.36 (**The Inverse of a Complex Line Bundle).** Let $\mathbb{E} \to M$ be a complex line bundle with transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then there is a complex line bundle $E^{-1} \to M$, unique up to isomorphism, with transition maps $g_{\beta\alpha}^{-1} : U_\alpha \cap U_\beta \to \mathbb{C}^*$. Its tensor product with E is isomorphic to the trivial bundle. Hence it follows from equation 10.56 that

$$c_1(E^{-1}) = -c_1(E).$$

Example 10.37 (**Complex Line Bundles over $\mathbb{C}\mathrm{P}^n$).** For $d \in \mathbb{Z}$ consider the complex line bundle

$$H^d := \frac{S^{2n+1} \times \mathbb{C}}{S^1} \to \mathbb{C}\mathrm{P}^n, \quad [z_0 : z_1 : \cdots : z_n; \zeta] \equiv [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n; \lambda^d \zeta].$$

For $d = 0$ this is the trivial bundle, for $d > 0$ it is the d-fold tensor product of the line bundle $H \to \mathbb{C}\mathrm{P}^n$ in Theorem 9.51, and we have $H^{-d} \cong (H^d)^{-1}$. Hence, by Theorem 9.51, equation (10.56), and Example 10.36, we have

$$c_1(H^d) = dh$$

for every $d \in \mathbb{Z}$. Here $h \in H^2(\mathbb{C}\mathrm{P}^n)$ is the positive integral generator with integral one over the submanifold $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^n$.

10.5 Chern Classes in Geometry

10.5.1 Complex Manifolds

A **complex n-manifold** is a real $2n$-dimensional manifold X equipped with an atlas $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that the transition maps

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic maps between open subsets of $\mathbb{C}^n$. This means that the real derivative of $\phi_\beta \circ \phi_\alpha^{-1}$ at every point is given by multiplication with a complex $n \times n$-matrix. A complex 1-manifold is called a **complex curve** and a complex 2-manifold is called a **complex surface**. Thus a complex curve has real dimension two and a complex surface has real dimension four. Complex manifolds are always oriented and their tangent bundles inherit complex structures from the coordinate charts. Thus the tangent bundle TX of a complex manifold has Chern classes. If X is a complex n-manifold with an atlas as above, a smooth function $f : U \to \mathbb{C}$ on an open subset $U \subset X$ is called **holomorphic** if $f \circ \phi_\alpha^{-1} : \phi_\alpha(U \cap U_\alpha) \to \mathbb{C}$ is holomorphic for each α. Equivalently, the derivative $df(p) : T_pX \to \mathbb{C}$ is complex linear for every $p \in U$.

Example 10.38 (The Chern Class of $\mathbb{C}\mathrm{P}^n$). The complex projective space $\mathbb{C}\mathrm{P}^n$ is a complex manifold and hence its tangent bundle has Chern classes. In the geometric description of $\mathbb{C}\mathrm{P}^n$ as the space of complex lines in $\mathbb{C}^{n+1}$ the tangent space of $\mathbb{C}\mathrm{P}^n$ at a point $\ell \in \mathbb{C}\mathrm{P}^n$ is given by

$$T_\ell\mathbb{C}\mathrm{P}^n = \mathrm{Hom}^{\mathbb{C}}(\ell, \ell^\perp).$$

Geometrically, every line in $\mathbb{C}^{n+1}$ sufficiently close to ℓ is the graph of a complex linear map from ℓ to $\ell^\perp$. Moreover, each complex linear map from ℓ to itself is given by multiplication with a complex number. In other words, $\mathrm{Hom}^{\mathbb{C}}(\ell, \ell) = \mathbb{C}$ and so there is an isomorphism

$$T_\ell\mathbb{C}\mathrm{P}^n \oplus \mathbb{C} \cong \mathrm{Hom}^{\mathbb{C}}(\ell, \ell^\perp \oplus \ell) = \mathrm{Hom}^{\mathbb{C}}(\ell, \mathbb{C}^{n+1})$$

Thus the direct sum of $T\mathbb{C}\mathrm{P}^n$ with the trivial bundle $H^0 = \mathbb{C}\mathrm{P}^n \times \mathbb{C}$ is the $(n+1)$-fold direct sum of the bundle $H \to \mathbb{C}\mathrm{P}^n$ in Theorem 9.51 with itself:

$$T\mathbb{C}\mathrm{P}^n \oplus H^0 = \underbrace{H \oplus H \oplus \cdots \oplus H}_{n+1 \text{ times}}.$$

Since $c(H) = 1 + h$ it follows from the *(Zero)* and *(Sum)* axioms that

$$c(T\mathbb{C}\mathrm{P}^n) = (1 + h)^{n+1},$$

where $h \in H^2(\mathbb{C}\mathrm{P}^n)$ is the positive integral generator as in Theorem 9.51.

10.5.2 Holomorphic Line Bundles

A **holomorphic line bundle** over a complex manifold X is a complex line bundle $\pi : E \to X$ equipped with local trivializations such that the transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are holomorphic. A **holomorphic section** of such a holomorphic line bundle E is a section $s : X \to \mathbb{C}$ that, in the local trivializations, is represented by holomorphic functions $s_\alpha : U_\alpha \to \mathbb{C}$. This notion makes sense because the s_α are related by

$$s_\beta = g_{\beta\alpha} s_\alpha$$

on $U_\alpha \cap U_\beta$ and the $g_{\beta\alpha}$ are holomorphic. If we choose a Hermitian structure on a holomorphic line bundle and Hermitian trivializations, the transition maps will no longer be holomorphic, by the maximum principle, unless they are locally constant. It is therefore often more convenient to use the original holomorphic trivializations.

Example 10.39 (Holomorphic Line Bundles over $\mathbb{C}\mathrm{P}^n$). The line bundle $H^d \to \mathbb{C}\mathrm{P}^n$ in Example 10.37 admits the structure of a holomorphic line bundle. More precisely, the standard atlas $\phi_i : U_i \to \mathbb{C}^n$ defined by

$$U_i := \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\}$$

and

$$\phi_i([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)$$

has holomorphic transition maps. A trivialization of H^d over U_i is the map $\psi_i : H^d|_{U_i} \to U_i \times \mathbb{C}$ defined by

$$\psi_i([z_0 : \cdots : z_n; \zeta]) := \left([z_0 : \cdots : z_n], \frac{\zeta}{z_i^d}\right).$$

The transition maps $g_{ji} : U_i \cap U_j \to \mathbb{C}^*$ are then given by

$$g_{ji}([z_0 : \cdots : z_n]) = \left(\frac{z_i}{z_j}\right)^d$$

and they are evidently holomorphic. For $d \geq 0$ every homogeneous complex polynomial $p : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d determines a holomorphic section

$$s([z_0 : \cdots : z_n]) = [z_0 : \cdots : z_n; p(z_0, \ldots, z_n)]$$

of H^d. It turns out that these are all the holomorphic sections of H^d and that the only holomorphic section of H^d for $d < 0$ is the zero section. However the proof of these facts would take us too far afield into the realm of algebraic geometry. An excellent reference is the book [4] by Griffiths and Harris.

10.5.3 The Adjunction Formula

Let X be a compact connected complex surface and

$$C \subset X$$

be a **smooth complex curve**. Thus C is a compact submanifold without boundary whose tangent space T_xC at each point $x \in C$ is a one dimensional complex subspace of T_xX. In particular, C is a compact oriented 2-manifold without boundary. The **adjunction formula** asserts

$$\langle c_1(TX), C\rangle = \chi(C) + C \cdot C, \tag{10.57}$$

where $C \cdot C$ denotes the self-intersection number of C, $\chi(C)$ denotes the Euler characteristic of C, and $\langle c_1(TX), C\rangle$ denotes the integral of (a representative of) the first Chern class $c_1(TX) \in H^2(X)$ over C.

To prove the adjunction formula we choose a Riemannian metric on X such that the complex structure on each tangent space T_xX is a skew symmetric automorphism. Thus both the tangent bundle of C and the normal bundle $TC^\perp$ are complex vector bundles over C and the restriction of TX to C is the direct sum

$$TX|_C = TC \oplus TC^\perp.$$

By the *(Euler Class)* axiom for the Chern classes and Example 9.45 we have

$$\langle c_1(TC), C\rangle = \langle e(TC), C\rangle = \chi(C).$$

Using the *(Euler Class)* axiom again we obtain

$$\langle c_1(TC^\perp), C\rangle = \langle e(TC^\perp), C\rangle = C \cdot C,$$

where the last equation follows from Theorem 9.42. Here we use the diffeomorphism

$$\exp : TC^\perp_\varepsilon \to U_\varepsilon$$

in (9.18) to identity the self-intersection number of the zero section in $TC^\perp$ with the self-intersection number of C in X. Now it follows from the *(Sum)* axiom for the Chern classes that

$$\langle c_1(TX), C\rangle = \langle c_1(TC), C\rangle + \langle c_1(TC^\perp), C\rangle$$

and this proves (10.57).

Now suppose that $\pi : E \to X$ is a holomorphic line bundle over a compact connected complex surface without boundary and $s : X \to E$ is a holomorphic section that is transverse to the zero section. Then it follows directly from the definitions that its zero set

$$C := s^{-1}(0)$$

is a compact complex curve without boundary. Let us also assume that C is connected and denote by g the genus of C, understood as a compact connected oriented 2-manifold without boundary. By Example 8.51 we have

$$\chi(C) = 2 - 2g$$

and hence the adjunction formula (10.57) takes the form

$$\begin{aligned} 2 - 2g &= \langle c_1(TX), C\rangle - C \cdot C \\ &= \langle c_1(TX) - c_1(E), C\rangle \\ &= \int_X \bigl(c_1(TX) - c_1(E)\bigr) \cup c_1(E) \end{aligned} \tag{10.58}$$

Here the second equality follows from the fact that the vertical differential Ds along $C = s^{-1}(0)$ furnishes an isomorphism form the normal bundle $TC^\perp$ to the restriction $E|_C$. The last equality follows from the fact that the Euler class $c_1(E) = e(E)$ is dual to C, by Theorem 9.47.

Example 10.40 (**Degree-d Curves in $\mathbb{C}\mathrm{P}^2$**)**.** As a specific example we take

$$X = \mathbb{C}\mathrm{P}^2, \qquad E = H^d.$$

Suppose that $p : \mathbb{C}^3 \to \mathbb{C}$ is a homogeneous complex degree-d polynomial and that the resulting holomorphic section $s : \mathbb{C}\mathrm{P}^2 \to H^d$ is transverse to the zero section (see Example 10.39). Then the zero set of s is a smooth degree-d curve

$$C_d = \bigl\{[z_0 : z_1 : z_2] \in \mathbb{C}\mathrm{P}^2 \,|\, p(z_0, z_1, z_2) = 0\bigr\}.$$

By Example 10.37 we have $c_1(H^d) = dh$ and by Example 10.38 we have $c_1(T\mathbb{C}\mathrm{P}^2) = 3h$. Thus equation (10.58) asserts that the genus $g = g(C_d)$ of the complex curve C_d satisfies the equation

$$2 - 2g = (3 - d)d \int_{\mathbb{C}\mathrm{P}^2} h \cup h = 3d - d^2.$$

Here the second equality follows from (9.33). Thus we have proved that

$$g(C_d) = \frac{(d-1)(d-2)}{2}. \tag{10.59}$$

This is the original version of the adjunction formula. One can verify it geometrically by deforming a degree-d curve to a union of d *generic* lines in $\mathbb{C}P^2$. Any two of these lines intersect in exactly one point and *"generic"* means here that these points are pairwise distinct. Thus we end up with a total of $d(d-1)/2$ intersection points. Performing a *connected sum* operation at each of the intersection points one can verify the formula (10.59).

A compact connected oriented 2-dimensional submanifold $\Sigma \subset \mathbb{C}P^2$ without boundary is said to **represent the cohomology class** dh if $dh = [\tau_\Sigma]$ is dual to Σ as in Section 8.4.3. Thus our complex degree-d curve C_d is such a representative of the class dh. A remarkable fact is that every representative of the class dh has at least the genus of C_d:

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}. \tag{10.60}$$

This is the socalled **Thom Conjecture** which was open for many years and was finally settled in the nineties by Kronheimer and Mrowka [8], using Donaldson theory. They later extended their result to much greater generality and proved, with the help of Seiberg–Witten theory, that every 2-dimensional symplectic submanifold with nonnegative self-intersection number in a symplectic 4-manifold minimizes the genus in its cohomology class. For an exposition see their book [9]. The case of negative self-intersection number was later settled by Ozsvath and Szabo [15].

10.5.4 Chern Class and Self-Intersection

Let X be a complex surface and $\Sigma \subset X$ be a compact oriented 2-dimensional submanifold without boundary. Then the integral of the first Chern class of TX over Σ agrees modulo two with the self-intersection number:

$$\langle c_1(TX), \Sigma \rangle \equiv \Sigma \cdot \Sigma \mod 2. \tag{10.61}$$

To see this, choose any complex structure on each of the real rank-2 bundles $T\Sigma$ and $T\Sigma^\perp$. Then the same argument as in the proof of the adjunction formula (10.57) shows that the integral of the first Chern class of this new complex structure on $TX|_\Sigma$ over Σ is the sum $\chi(\Sigma) + \Sigma \cdot \Sigma$. Since the Euler characteristic $\chi(\Sigma)$ is even and the integrals of the first Chern classes of $TX|_\Sigma$ with both complex structures agree modulo two, by Exercise 10.41 below, this proves (10.61).

Exercise 10.41 (Complex Rank-2 Bundles over Real 2-Manifolds). Let Σ be compact connected oriented 2-manifold without boundary.

(i) There are precisely two oriented real rank 4-bundles over Σ, one trivial and one nontrivial.

(ii) Every oriented real rank 4-bundle admits a complex structure compatible with the orientation.

(iii) A complex rank-2-bundle $\pi : E \to \Sigma$ admits a real trivialization if and only if its first Chern number $\langle c_1(E), \Sigma\rangle = \int_\Sigma c_1(E)$ is even.

Hint 1: An elegant proof of these facts can be given by means of the Stiefel–Whitney classes (see Milnor–Stasheff [12]).

Hint 2: Consider the trivial bundle $\Sigma \times \mathbb{R}^4$ and identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ via $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ where $\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. Show that every complex structure on $\mathbb{H}$ that is compatible with the inner product and orientation has the form

$$J_\lambda = \lambda_1 \mathbf{i} + \lambda_2 \mathbf{j} + \lambda_3 \mathbf{k}, \qquad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1.$$

Thus a complex structure on $E = \Sigma \times \mathbb{H}$ that is compatible with the metric and orientation has the form $z \mapsto J_{\lambda(z)}$ where $\lambda : \Sigma \to S^2$ is a smooth map. Prove that the first Chern number of (E, J_λ) is given by

$$\int_\Sigma c_1(E, J_\lambda) = 2 \deg(\lambda : \Sigma \to S^2).$$

Use the ideas in the next Hint.

Hint 3: Here is a sketch of a proof that the first Chern numbers of any two complex structures on an oriented real rank 4-bundle $\pi : E \to \Sigma$ agree modulo two. By transversality every real vector bundle whose rank is bigger than the dimension of the base has a nonvanishing section (see Chapter 5). Hence E has two linearly independent sections s_1 and s_2. Denote by $\Lambda \subset E$ the subbundle spanned by s_1 and s_2. Given a complex structure J on E denote by $E_1 \subset E$ the complex subbundle spanned by s_1 and Js_1. Thus E_1 has a global trivialization and so the first Chern number of the complex line bundle E/E_1 agrees with the first Chern number of (E, J). Show that this number agrees modulo two with the Euler number of the oriented real rank-2 bundle E/Λ. To see this, think of s_2 as a section of E/E_1 and of Js_1 as a section of E/Λ. Both sections have the same zeros: the points $z \in \Sigma$ where Λ_z is a complex subspace of E_z. Prove that the transversality conditions for both sections are equivalent. Compare the indices of the zeros.

Hint 4: Choose an closed disc $D \subset \Sigma$ and show via parallel transport that the restrictions of E to both D and $\overline{\Sigma \setminus D}$ admit global trivializations. This requires the existence of a pair-of-pants decomposition of Σ (see Hirsch [6]). Assuming this we obtain two trivializations over the boundary $\Gamma := \partial D \cong S^1$. These differ by a loop in the structure group. In the complex case this construction gives rise to a loop $g : S^1 \to \mathrm{U}(2) \subset \mathrm{SO}(4)$. In the real case we get a loop in $\mathrm{SO}(4)$. Prove that, in the complex case with the appropriate choice of orientations, the first Chern number of E is given by

$$\int_\Sigma c_1(E) = \deg(\det \circ g : S^1 \to S^1).$$

Prove that a loop $g : S^1 \to \mathrm{U}(2)$ is contractible in $\mathrm{SO}(4)$ if and only if the degree of the composition $\det \circ g : S^1 \to S^1$ has even degree.

10.5.5 The Hirzebruch Signature Theorem

Let X be a compact connected oriented smooth 4-manifold without boundary. Then Poincaré duality (Theorem 8.38) asserts that the Poincaré pairing

$$H^2(X) \times H^2(X) \to \mathbb{R} : ([\omega], [\tau]) \mapsto \int_X \omega \wedge \tau, \tag{10.62}$$

is nondegenerate. The pairing (10.62) is a symmetric bilinear form, also called the **intersection form of** X and denoted by

$$Q_X : H^2(X) \times H^2(X) \to \mathbb{R}.$$

Thus the second Betti number $b_2(X) = dim H^2(X)$ is a sum

$$b_2(X) = b^+(X) + b^-(X)$$

where $b^+(X)$ is the maximal dimension of a subspace of $H^2(X)$ on which the intersection form Q_X is positive definite and $b^-(X)$ is the maximal dimension of a subspace of $H^2(X)$ on which Q_X is negative definite. Equivalently, $b^+(X)$ is the number of positive entries and $b^-(X)$ is the number of negative entries in any diagonalization of Q_X. The **signature** of X is defined by

$$\sigma(X) := b^+(X) - b^-(X).$$

The **Hirzebruch Signature Theorem** asserts that, if X is a complex surface, then

$$\int_X c_1(TX) \cup c_1(TX) = 2\chi(X) + 3\sigma(X). \tag{10.63}$$

Equivalently, the signature is one third of the integral of the cohomology class $c_1(TX)^2 - 2c_2(TX) \in H^4(X)$ over X. The class $c_1^2 - 2c_2$ is the first Pontryagin class and is also defined for arbitrary real vector bundles $E \to X$ (see Exercise 10.35). Thus equation (10.63) can be expressed in the form

$$\sigma(X) = \frac{1}{3} p_1(TX).$$

(Here we use the same notation $p_1(TX)$ for a 4-dimensional deRham cohomology class and for its integral over X.) In this form the Hirzebruch Signature Theorem remains valid for all compact connected oriented smooth 4-manifold without boundary. It is a deep theorem in differential topology and its proof goes beyond the scope of this manuscript.

As an explicit example consider the complex projective plane

$$X = \mathbb{C}\mathrm{P}^2, \qquad c_1(X) = 3h, \qquad \chi(X) = 3, \qquad \sigma(X) = 1,$$

Another example is

$$X = S^2 \times S^2, \qquad c_1(X) = 2a + 2b, \qquad \chi(X) = 4, \qquad \sigma(X) = 0.$$

Here we choose as a basis of $H^2(S^2 \times S^2)$ the cohomology classes a and b of two volume forms with integral one on the two factors, pulled back to the product. The intersection form is in this basis given by

$$Q_X \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A third example is the 4-torus $X = \mathbb{T}^4 = \mathbb{C}^2/\mathbb{Z}^4$ with its standard complex structure. In this case both Chern classes are zero and $\chi(\mathbb{T}^4) = \sigma(\mathbb{T}^4) = 0$.
Exercise: Verify the last equality by choosing a suitable basis of $H^2(\mathbb{T}^4)$. Verify the Hirzebruch signature formula in all three cases.

10.5.6 Hypersurfaces of $\mathbb{C}\mathrm{P}^3$

An interesting class of complex 4-manifolds is given by complex hypersurfaces of $\mathbb{C}\mathrm{P}^3$. More precisely, consider the holomorphic line bundle $H^d \to \mathbb{C}\mathrm{P}^3$ in Example 10.39, let $p : \mathbb{C}^4 \to \mathbb{C}$ be a homogeneous complex degree-d polynomial, and assume that the resulting holomorphic section $s : \mathbb{C}\mathrm{P}^3 \to H^d$ is transverse to the zero section. Denote the zero set of s by

$$X_d := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,|\, p(z_0, z_1, z_2, z_3) = 0 \right\}.$$

This is a complex submanifold of $\mathbb{C}P^3$ and hence is a complex surface. in this case the **Hard Lefschetz Theorem** asserts that X_d is connected and simply connected. (More generally, the Hard Lefschetz Theorem asserts that the zero set of a transverse holomorphic section of a *"sufficiently nice"* holomorphic line bundle inherits the homotopy and cohomology groups of the ambient manifold below the middle dimension; *"nice"* means that the line bundle has lots of holomorphic sections or, in technical terms, is *"ample"*. The holomorphic line bundle $H^d \to \mathbb{C}P^n$ satisfies this condition for $d > 0$.) We prove that

$$\begin{aligned} \chi(X_d) &= d^3 - 4d^2 + 6d, \\ \sigma(X_d) &= \frac{4d - d^3}{3}, \\ b^+(X_d) &= \frac{d^3 - 6d^2 + 11d - 3}{3}, \\ b^-(X_d) &= \frac{2d^3 - 6d^2 + 7d - 3}{3}. \end{aligned} \tag{10.64}$$

To see this, we first observe that, by Poincaré duality and the the Hard Lefschetz theorem, we have $b_0(X_d) = b_4(X_d) = 1$ and $b_1(X_d) = b_3(X_d) = 0$. Hence

$$\chi(X_d) = 2 + b^+ + b^-$$

and so the last two equations in (10.64) follow from the first two. Next we choose a Riemannian metric on $\mathbb{C}P^3$ with respect to which the standard complex structure is skew-symmetric (for example the Fubini–Study metric [4]). This gives a splitting

$$T\mathbb{C}P^3|_{X_d} = TX_d \oplus TX_d^\perp$$

into complex subbundles. The vertical differential of s along X again provides us with an isomorphism $Ds : TX_d^\perp \to E|_{X_d}$. Thus, by the *(Sum)* axiom for the Chern classes and Example 10.38, we have

$$(1 + h)^4 = c(T\mathbb{C}P^3) = c(TX_d)c(TX_d^\perp) = c(TX_d)(1 + dh).$$

Here we think of the cohomology classes on $\mathbb{C}P^3$ as their restrictions to X_d. Abbreviating $c_1 := c_1(TX_d)$ and $c_2 := c_2(TX_d)$ we obtain

$$1 + 4h + 6h^2 = (1 + c_1 + c_2)(1 + dh) = 1 + (c_1 + dh) + (c_2 + dhc_1)$$

and hence

$$c_1 = (4 - d)h, \qquad c_2 = 6h^2 - dhc_1 = (d^2 - 4d + 6)h^2.$$

Since X_d is the zero set of a smooth section of H^d it is dual to the Euler class $e(H^d) = c_1(H^d) = dh$ (see Example 10.37), by Theorem 9.47. Hence

$$\int_{X_d} h \cup h = d \int_{\mathbb{C}\mathrm{P}^3} h \cup h \cup h = d.$$

Here the second equality follows from (9.33). Combining the last three equations we find

$$\chi(X_d) = \int_{X_d} c_2(TX) = (d^2 - 4d + 6) \int_{X_d} h \cup h = d^3 - 4d^2 + 6d$$

and

$$\int_{X_d} c_1(TX_d) \cup c_1(X_d) = (d-4)^2 \int_{X_d} h \cup h = d(d-4)^2.$$

Hence the Hirzebruch signature formula gives

$$\sigma(X_d) = \frac{d(d-4)^2 - 2d^3 + 8d^2 - 12d}{3} = \frac{4d - d^3}{3}$$

and this proves (10.64).

The first two examples are $X_1 \cong \mathbb{C}\mathrm{P}^2$ and $X_2 \cong S^2 \times S^2$. The reader may verify that the numbers in equation (10.64) match in these cases. The cubic surfaces in $\mathbb{C}\mathrm{P}^3$ are all diffeomorphic to $\mathbb{C}\mathrm{P}^2$ with six points *blown up.* This blowup construction is an operation in algebraic geometry, where one removes a point in the manifold and replaces it by the set of all complex lines through the origin in the tangent space at that point. Such a blowup admits in a canonical way the structure of a complex manifold [4]. An alternative description of X_3 is as a connected sum

$$X_3 = \mathbb{C}\mathrm{P}^2 \# 6\overline{\mathbb{C}\mathrm{P}}^2.$$

Here $\overline{\mathbb{C}\mathrm{P}}^2$ refers to the complex projective plane with the orientation reversed, which is not a complex manifold. (Its signature is minus one and the number $2\chi(\overline{\mathbb{C}\mathrm{P}}^2) + 3\sigma(\overline{\mathbb{C}\mathrm{P}}^2) = 3$ is not the integral of the square of any 2-dimensional cohomology class.) The symbol # refers to the connected sum operation where one cuts out balls from the two manifolds and glues the complements together along their boundaries, which are diffeomorphic to the 3-sphere. The resulting manifold is oriented and the numbers $b^{\pm}$ are additive under this operation Thus

$$\chi(X_3) = 9, \qquad \sigma(X_3) = -5, \qquad b^+(X_3) = 1, \qquad b^-(X_3) = 6$$

and this coincides with (10.64) for $d = 3$.

Particularly interesting examples are the quartic surfaces in $\mathbb{C}\mathrm{P}^3$. They are **K3-surfaces**. These can be uniquely characterized (up to diffeomorphism) as compact connected simply connected complex surfaces without boundary whose first Chern classes vanish. These manifolds do not all admit complex embeddings into $\mathbb{C}\mathrm{P}^3$ but the surfaces of type X_4 are examples. They have characteristic numbers

$$\chi(X_4) = 24, \qquad \sigma(X_4) = -16, \qquad b^+(X_4) = 3, \qquad b^-(X_4) = 19,$$

which one can read off equation (10.64). One can also deduce these numbers from the Hirzebruch signature formula, which in this case takes the form $0 = 2\chi + 3\sigma = 4 + 5b^+ - b^-$. That the number b^+ must be equal to 3 follows from the existence of a Ricci-flat Kähler metric, a deep theorem of Yau, and this implies that the complex exterior power $\Lambda^{2,0}T^*X$ has a global nonvanishing holomorphic section. Therefore the dimension p_g of the space of holomorphic sections of this bundle is equal to one, and it then follows from Hodge theory that $b^+ = 1 + 2p_g = 3$. The details of this lie again much beyond what is covered in the present manuscript.

The distinction between the cases

$$d < 4, \qquad d = 4, \qquad d > 4$$

for hypersurfaces of $\mathbb{C}\mathrm{P}^3$ is analogous to the distinction of complex curves in terms of the genus. For curves in $\mathbb{C}\mathrm{P}^2$ these are the cases $d < 3$ (genus zero/positive curvature), $d = 3$ (genus one/zero curvature), and $d > 3$ (higher genus/negative curvature). In the present situation the case $d < 4$ gives examples of Fano surfaces analogous to the 2-sphere, the $K3$-surfaces with $d = 4$ correspond to the 2-torus allthough they do not admit flat metrics, and for $d > 4$ the manifold X_d is an example of a **surface of general type** in analogy with higher genus curves.

Exercise 10.42. Show that the polynomial $p(z_0, \dots, z_n) = z_0^d + \cdots + z_n^d$ on $\mathbb{C}^{n+1}$ gives rise to a holomorphic section $s : \mathbb{C}\mathrm{P}^n \to H^d$ that is transverse to the zero section. Hence its zero set X_d is a smooth complex hypersurface of $\mathbb{C}\mathrm{P}^n$. Prove that its first Chern class is zero whenever $d = n + 1$. Kähler manifolds with this property are called **Calabi–Yau manifolds**. The $K3$-surfaces are examples. The quintic hypersurfaces of $\mathbb{C}\mathrm{P}^4$ are examples of Calabi–Yau 3-folds and they play an important role in geometry and physics.

Exercise 10.43. Compute the Betti numbers of a degree-d hypersurface in $\mathbb{C}\mathrm{P}^4$. **Hint:** The Hard Lefschetz Theorem asserts in this case that $b_0(X_d) = b_2(X_d) = 1$ and $b_1(X_d) = 0$.

10.5.7 Almost complex structures on 4-manifolds

Let X be an oriented $2n$-manifold. An **almost complex structure** on X is an automorphism of the tangent bundle TX with square minus one:

$$J : TX \to TX, \qquad J^2 = -\mathbb{1}.$$

The tangent bundle of any complex manifold has such a structure, as the multiplication by $\mathbf{i} = \sqrt{-1}$ in the coordinate charts carries over to the tangent bundle. However, not every almost complex structure arises from a complex structure (except in real dimension two).

Let us now assume that X is a compact connected oriented smooth 4-manifold without boundary. Let J be an almost complex structure on X and denote its first Chern class in deRham cohomology by

$$c := c_1(TX, J) \in H^2(X).$$

This is an integral class in that the number $c \cdot \Sigma = \langle c, \Sigma \rangle = \int_\Sigma c$ is an integer for every compact oriented 2-dimensional submanifold $\Sigma \subset X$. Moreover, equation (10.61) carries over to the almost complex setting so that

$$c \cdot \Sigma \equiv \Sigma \cdot \Sigma \mod 2 \tag{10.65}$$

for every Σ as above. The Hirzebruch signature formula also continues to hold in the almost complex setting and hence

$$c^2 = 2\chi(X) + 3\sigma(X). \tag{10.66}$$

Here we abbreviate $c^2 := \langle c^2, X \rangle = \int_X c^2 \in \mathbb{Z}$. It turns out that, conversely, for every integral deRham cohomology class $c \in H^2(X)$ that satisfies (10.65) and (10.66) there is an almost complex structure J on X with $c_1(TX, J) = c$. We will not prove this here. However, this can be used to examine which 4-manifolds admit almost complex structures and to understand their first Chern classes.

Exercise 10.44. Consider the 4-manifold $X = \mathbb{C}\mathrm{P}^2 \# k \overline{\mathbb{C}\mathrm{P}}^2$ (the projective plane with k points blown up). This manifold admits a complex structure by a direct construction in algebraic geometry [4]. Verify that it admits an almost complex structure by finding all integral classes $c \in H^2(X)$ that satisfy (10.65) and (10.66). Start with $k = 0, 1, 2$.

Exercise 10.45. The k-fold connected sum $X = k\mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \cdots \# \mathbb{C}\mathrm{P}^2$ admits an almost complex structure if and only if k is odd.

Exercise 10.46. Which integral class $c \in H^2(\mathbb{T}^4)$ is the first Chern class of an almost complex structure on $\mathbb{T}^4$.

10.6 Low Dimensional Manifolds

The examples in the previous section show that there is a rich world of manifolds out there whose study is the subject of differential topology and other related areas of mathematics, including complex, symplectic, and algebraic topology. The present notes only scratch the surface of some of these areas. One fundamental question in differential topology is how to tell if two manifolds of the same dimension m are diffeomorphic, or perhaps not diffeomorphic as the case may be. In this closing section we discuss some classical and more recent answers to this question.

The easiest case is of course $m = 1$. We have proved in Chapter 2 that every compact connected smooth 1-manifold *without boundary* is diffeomorphic to the circle and in the case of *nonempty boundary* is diffeomorphic to the closed unit interval. We have seen that this observation plays a central role in the definitions of degree and intersection number, and in fact throughout differential topology.

The next case is $m = 2$, where this question is also completely understood, although the proof is considerably harder. Two compact connected oriented smooth 2-manifolds without boundary are diffeomorphic if and only if they have the same genus. As pointed out in Example 8.51, a beautiful proof of this theorem, based on Morse theory, is contained in the book of Hirsch [6]. The result generalizes to all compact 2-manifolds with or without boundary, and orientable or not. Both in the orientable and in the nonorientable case the diffeomorphism type of a compact connected 2-manifold is determined by the Euler characteristic and the number of boundary components. The proof is also contained in [6]. This does not mean, however, that the study of 2-manifolds has now been settled. For example the study of real 2-manifolds equipped with complex structures (called **Riemann surfaces**) is a rich field of research with connections to many areas of mathematics such as algebraic geometry, number theory, and dynamical systems. A classical result is the **uniformization theorem**, which asserts that every connected simply connected Riemann surface is holomorphically diffeomorphic to either the complex plane, or the open unit disc in the complex plane, or the 2-sphere with its standard complex structure. In particular, it is not necessary to assume that the Riemann surface is paracompact; paracompactness is a consequence of uniformization. This is a partial answer to a complex analogue of the aforementioned question. We remark that interesting objects associated to oriented 2-manifolds are, for example, the mapping class group (diffeomorphisms up to isotopy) and Teichmüller space (complex structures up to diffeomorphisms isotopic to the identity).

The compact connected manifolds without boundary in dimensions one and two are not simply connected except for the 2-sphere. Let us now turn to the higher dimensional case and focus on simply connected manifolds. In dimension three a central question, which was open for about a century, is the following.

Three Dimensional Poincaré Conjecture. *Every compact connected simply connected* 3*-manifold* M *without boundary is diffeomorphic to* S^3*.*

This conjecture has recently (in the early years of the 21st century) been confirmed by Grigory Perelman. His proof is a modification of an earlier program by Richard Hamilton to use the socalled **Ricci flow** on the space of all Riemannian metrics on M. The idea is, roughly speaking, to start with an arbitrary Riemannian metric and use it as an initial condition for the Ricci flow. It is then a hard problem in geometry and nonlinear parabolic partial differential equations to understand the behavior of the metric under this flow. The upshot is that, through lot of hard analysis and deep geometric insight, Perelman succeded in proving that the flow does converge to a round metric (with constant sectional curvature). Then a standard result in differential geometry provides a diffeomorphism to the 3-sphere. The proof of the Poincaré conjecture is one of the deepest theorems in differential topology and is an example of the power of analytical tools to settle questions in geometry and topology. There are now many expositions of Perelman's proof of the three dimensional Poincaré conjecture, beyond Perelman's original papers, too numerous to discuss here. An example is the detailed book by Morgan and Tian [13].

The higher dimensional analogue of the the Poincaré conjecture asserts that every compact connected simply connected smooth m-manifold M without boundary whose integral cohomology is isomorphic to that of the m-sphere, i.e.

$$H^k(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } k = 0 \text{ and } m, \\ 0, & \text{for } 1 \le k \le m-1, \end{cases}$$

is diffeomorphic to the m-sphere. This question is still open in dimension four. However, by the work of Michael Freedman, it is known that every such 4-manifold is homeomorphic to the the 4-sphere. In fact one distingushes between the **smooth Poincaré conjecture** (which asserts the existence of a diffeomorphism) and the **topological Poincaré conjecture** (which asserts the existence of a homeomorphism). Remarkably, the higher dimensional Poincaré conjecture is much easier to understand than in dimensions three and four. It was settled long ago by Stephen Smale with the methods of Morse theory. A beautiful exposition is Milnor's book [11]. The topological Poincaré conjecture holds in all dimensions $m \geq 5$. But in certain dimensions there are socalled **exotic spheres** that are homeomorphic but not diffeomorphic to the m-sphere. Examples are Milnor's famous exotic 7-spheres. Later work by Kervaire and Milnor showed that there are precisely 27 exotic spheres in dimension seven.

Let us now turn to compact connected simply connected smooth 4-manifolds X without boundary and with $H^2(X) \neq 0$. The intersection form

$$Q_X : H^2(X) \times H^2(X) \to \mathbb{R}$$

is then a diffeomorphism invariant and so are the numbers $b^+(X)$ and $b^-(X)$ (see Section 10.5.5). They are determined by the Euler characteristic and signature of X. In fact, more is true. The intersection form can be defined on integral cohomology and Poincaré duality over the integers asserts that it remains nondegenerate over the integers (which can be proved with the same methods as Theorem 8.38 once an integral cohomology theory has been set up). This means that it is represented by a symmetric integer matrix with determinant ± 1 in any integral basis of $H^2(X;\mathbb{Z})$.

This leads to the issue of understanding quadratic forms over the integers. One must distinguish between the even and odd case, where **even** means that $Q(a,a)$ is even for every integer vector a and **odd** means that $Q(a,a)$ is odd for some integer vector a. Thus an oriented 4-manifold X is called **even** if the self-intersection number of every compact oriented 2-dimensional submanifold $\Sigma \subset X$ without boundary is even and it is called **odd** is the self-intersection number is odd for some Σ. This property (being even or odd) is called the **parity of** X. For example, it follows from the formula (10.61) that a hypersurface $X_d \subset \mathbb{C}\mathrm{P}^3$ of degree d is odd if and only if d is odd. (Exercise: Prove this using the fact that $c_1(X_d) = (4-d)h$. Find a surface with odd self-intersection number when d is odd.)

Examples of even quadratic forms are

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_8 := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Both matrices are symmetric and have determinant ± 1. The second matrix is the Cartan matrix associated to the Dynkin diagram E_8 and is positive definite. A quadratic form (over the integers) is called **indefinite** if both b^+ and b^- are nonzero. The classification theorem for nondegenerate quadratic forms over the integers asserts that every indefinite nondegenerate quadratic form is diagonalizable over the integers in the odd case (with entries ± 1 on the diagonal) and in the even case is isomorphic to a direct sum of copies of H and $\pm E_8$. It follows, for example, that the self-intersection form of a $K3$-surface is isomorphic to $3H - 2E_8$. However, there are many positive (or negative) definite exotic quadratic forms. A deep theorem of Donaldson, that he proved in the early eighties, asserts that the intersection form of a smooth 4-manifold is diagonalizable, whenever it is positive or negative definite. Thus the exotic forms do not appear as intersection forms of smooth 4-manifolds.

Donaldson's Diagonalizability Theorem. *If X is a compact connected oriented smooth 4-manifold without boundary with definite intersection form Q_X, then Q_X is diagonalizable over the integers.*

Combining this with the aforementioned known facts about quadratic forms over the integers, we see that two compact connected simply connected oriented smooth 4-manifolds without boundary have isomorphic intersection forms over the integers if and only if they have the same Euler characteristic, signature, and parity. Now a deep theorem of Michael Freedman asserts that two compact connected simply connected oriented smooth 4-manifolds without boundary are homeomorphic if and only if they have isomorphic intersection forms over the integers. In the light of Donaldson's theorem Freedman's result can be rephrased as follows.

Freedman's Theorem. *Two compact connected simply connected oriented smooth 4-manifold without boundary are homeomorphic if and only if they have the same Euler characteristic, signature, and parity.*

A corollary is the Topological Poincaré Conjecture in Dimension Four. A natural question is if Freedman's theorem can be strengthened to provide a diffeomorphism. The answer is negative. In the early eighties, around the same time when Freedman proved his theorem, Donaldson discovered remarkable invariants of compact oriented smooth 4-manifolds without boundary by studying the anti-self-dual Yang–Mills equations with structure group SU(2). He proved that the resulting invariants are nontrivial for Kähler surfaces whereas they are trivial for every connected sum $X_1 \# X_2$ with $b^+(X_i) > 0$. Thus two such manifolds cannot be diffeomorphic.

Donaldson's Theorem. *Let X be a compact connected simply connected Kähler surface without boundary and assume*

$$b^+(X) \geq 2.$$

Then X is not diffeomorphic to any connected sum $k\mathbb{C}\mathrm{P}^2\#\ell\overline{\mathbb{C}\mathrm{P}}^2$.

If course, the only candidate for such a connected sum would be with

$$k = b^+(X), \qquad \ell = b^-(X).$$

This manifold has trivial Donaldson invariants because $k \geq 2$, and therefore cannot be diffeomorphic to X. To make the statement interesting we also have to assume that X is odd. Then the two manifolds are homeomorphic, by Freedman's theorem. An infinite sequence of examples is provided by hypersurfaces $X_d \subset \mathbb{C}\mathrm{P}^3$ of odd degree $d \geq 5$ (see Section 10.5.6). These are connected simply connected Kähler surfaces, satisfy $b^+(X_d) \geq 2$, and they are odd. Hence Donaldson's theorem applies, and Friedmans theorem furnishes a homeomorphism to a suitable connected sum of $\mathbb{C}\mathrm{P}^2$'s and $\overline{\mathbb{C}\mathrm{P}}^2$'s.

A beautiful introduction to Donaldson theory can be found in the book by Donaldson and Kronheimer [3]. The book includes a proof of Donaldson's Diagonalizability Theorem, which is also based on the study of anti-self-dual SU(2)-instantons. When Seiberg–Witten theory was discovered in the mid nineties, Taubes proved that all symplectic 4-manifolds have nontrivial Seiberg–Witten invariants. Since the Seiberg–Witten invariants of connected sums have the same vanishing properties as Donaldson invariants, this gave rise to an extension of Donaldson's theorem with the word *Kähler surface* replaced by *symplectic 4-manifold*. Both Donaldson theory and Seiberg–Witten theory are important topics in the study of 3- and 4-dimensional manifolds with a wealth of results in various directions, the Kronheimer–Mrowka proof of the Thom conjecture being just one example (see Section 10.5.3). In a nutshell one can think of these as intersection theories in suitable infninite dimensional settings. This shows again the power of analytical methods in topology and geometry.

Bibliography

[1] Ralph Abraham, Joel W. Robbin, *Transversal Mappings and Flows.* Benjamin Press, 1967.

[2] Raoul Bott, Loring W. Tu, *Differential Forms in Algebraic Topology.* Graduate Texts in Mathematics, **82**. Springer, 1982.

[3] Simon K. Donaldson, Peter B. Kronheimer *The Geometry of Four-Manifolds.* Oxford University Press, 1990.

[4] Phillip Griffiths, Joseph Harris, *Principles of Algebraic Geometry.* John Wiley & Sons, 1978.

[5] Victor Guillemin, Alan Pollack, *Differential Topology.* Prentice-Hall, 1974.

[6] Morris W. Hirsch, *Differential Topology.* Springer, 1994.

[7] John L. Kelley, *General Topology.* Graduate Texts in Mathematics, **27**. Springer, 1975.

[8] Peter B. Kronheimer, Tomasz Mrowka, Embedded surfaces and the structure of Donaldson's polynomial invariants. *J. Diff. Geom.* **41** (1995), 573734.

[9] Peter B. Kronheimer, Tomasz Mrowka, *Monopoles and Three-Manifolds.* Cambridge University Press 2008.

[10] John W. Milnor, *Topology from the Differentible Viewpoint.* The University Press of Vorginia, 1969.

[11] John W. Milnor, *Lectures on the h-Cobordism Theorem.* Princeton University Press, 1965.

[12] John W. Milnor, James D Stasheff *Characteristic Classes.* Princeton University Press, 1974.

[13] John W. Morgan, Gang Tian, *Ricci Flow and the Poincar Conjecture.* AMS, 2007.

[14] James R. Munkres, *Topology*, Second Edition. Prentice Hall, 2000.

[15] Peter Ozsvath, Zoltan Szabo, The symplectic Thom conjecture. *Ann. of Math.* **151** (2000), 93–124.

[16] Joel W. Robbin, Dietmar A. Salamon, *Introduction to Differential Geometry.* In preparation.

Index