4 Vector Bundles

Fibre bundles In 1934, Herbert Seifert published The *Topology of 3 - Dimensional Fibered Spaces*, which contained a definition of an object that is **a kind of fibre bundle**. Seifert was only considering circles as fibres and 3-manifolds for the total space.

Rational functions over $\mathbb{CP}^1$ We study function theory on domains in $\mathbb{C}$, on $\mathbb{C}$ and on the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$, or more generally on Riemann surface (1 dimensional complex manifold) and complex manifolds.

The simplest compact complex manifold is the Riemann sphere $\hat{\mathbb{C}} = \mathbb{CP}^1$. The following fact is well known by the maximum principle:

Theorem 4.1 *There is no non-constant holomorphic function on $\hat{\mathbb{C}}$.*

Nevertheless, there are lots of polynomials f

$$f := a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0, \quad a_n \neq 0 \tag{6}$$

defined over $\mathbb{C}$, which can be regarded as a meromorphic function on $\hat{\mathbb{C}}$ with the pole at the infinity.

Functions, graphs and sections of line bundles A function $f : \mathbb{C} \to \mathbb{C}$ can be regardrd as a graph:

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \times \mathbb{C} \\ z & \mapsto & (z, f(z)) \end{array}$$

so that

$$\{all\ functions\ f\ on\ \mathbb{C}\} \longleftrightarrow \{all\ the\ graphis\ of\ f\}.$$

In other words, we have a (trivial) line bundle $\pi : L := \mathbb{C} \times \mathbb{C} \to \mathbb{C}, (z, u) \mapsto z$ and f is a section of this bundle:

$$\begin{array}{c} L \\ \uparrow f \quad \downarrow \pi \\ \mathbb{C} \end{array} \tag{7}$$

$$\{all\ functions\ f\ on\ \mathbb{C}\} \longleftrightarrow \{all\ sections\ of\ the\ trivial\ line\ bundle\}.$$

In general, a *fiber bundle* is intuitively a space E which locally "looks" like a product space $B \times F$, but globally may have a different topological structure. More precsiely, a fiber bundle with fiber F is a map

$$\pi : E \to B$$

where E is called the *total space* of the fiber bundle and B the *base space* of the fiber bundle. The main condition for the map to be a fiber bundle is that every point in the base space $b \in B$ has a neighborhood U such that $f^{-1}(U)$ is homeomorphic to $U \times F$ in a special way. In particular if each $\pi^{-1}(x)$ is a vector space which changes smoothly, it is called a vector bundle. A Möbius band is the simplest non-trivial example of a vector bundle. If each $\pi^{-1}(x)$ is one dimensional vector space, it is called a line bundle. [14]

A *section* of a fiber bundle, $\pi : E \to B$, over a topological space, B, is a continuous map, $s : B \to E$, such that $\pi(s(x)) = x$ for all $x \in B$. (7) is a line bundle and

$$\{all\ functions\ defined\ on\ M\} \longleftrightarrow \{all\ sections\ of\ the\ trivial\ line\ bundle\ M \times \mathbb{C}\}$$

Vector bundles over a manifold Let M be a C^∞ differentiable manifold of dimension m and let $K = \mathbb{R}$ or $K = \mathbb{C}$ be the scalar field. A (real, complex) *vector bundle of rank r* over M is a C^∞ manifold E together with

i) a C^∞ map $\pi : E \to M$ which called the *projection*,

ii) a K-vector space structure of dimension r on each fiber $E_x = \pi^{-1}(x)$ such that the vector space structure is locally trivial. This means that there exists an open covering $\{V_a\}_{a \in I}$ of M and C^∞ diffeomorphisms called *trivializations*

$$\begin{array}{lccc} E & \supset & \pi^{-1}(V_\alpha) & \xrightarrow{\theta_\alpha \simeq} & V_\alpha \times K^r \\ \downarrow & & & & \\ X & \supset & V_\alpha & & \end{array}$$

such that for every $x \in V_\alpha$ the restriction map $\theta_\alpha(x) : \pi^{-1}(x) \to \{x\} \times K^r$ is a linear isomorphism.

Then for each $\alpha, \beta \in I$, the map

$$\begin{array}{rccc} \theta_{\alpha\beta} := \theta_\alpha \circ \theta_\beta^{-1} : & (V_\alpha \cap V_\beta) \times K^r & \to & (V_\alpha \cap V_\beta) \times K^r \\ & (x, \xi) & \mapsto & \big(x, g_{\alpha\beta}(x) \cdot \xi\big) \end{array} \tag{8}$$

where $\{g_{\alpha\beta}\}_{\alpha,\beta \in I}$ is a collection of invertible matrices with coefficients in $C^\infty(V_\alpha \cap V_\beta, K)$. They satisfy

$$\begin{cases} g_{\alpha\alpha} = Id, & on\ V_\alpha, \\ g_{\alpha\beta} g_{\beta\alpha} = Id, & on\ V_\alpha \cap V_\beta, \\ g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = Id, & on\ V_\alpha \cap V_\beta \cap V_\gamma. \end{cases} \tag{9}$$

[14] For more detailed definitions of holomorphic vector bundles and holomorphic line bundles, see [H05], p.66.

Such collection $\{g_{\alpha\beta}\}$ is called a system of transition matrices. Any collection of invertible matrices satisfying (9) defines a vector bundle E, obtained by gluing the charts $V_\alpha \times K^r$ via the identifications $\theta_{\alpha\beta}$.

When $r = 1$, a vector bundle is called a *line bundle* and transition matrices are called *transition functions*.

Let X be a complex manifold and E be a vector bundle over X with $K = \mathbb{C}$. Suppose that all $g_{\alpha\beta}$ as above are matrices whose entries are all holomorphic functions. Then E is called a *holomorphic vector bundle* over X. A holomorphic line bundle L over X is called a *line bundle* for simplicity.

Each vector bundle $\pi : E \to B$, we can define its *dual vector bundle* $\pi_* : E^* \to B$ whose fiber $\pi_*^{-1}(x)$ is the dual vector space of $\pi^{-1}(x)$for any point $x \in B$. [15] For any two vector bundles $\pi : E \to B$ and $\pi' : E' \to B$, we can define the *tensor product* $E \otimes E'$, still a vector bundle, over B, whose fiber is equal to the tensor product of vector spaces $\pi^{-1}(x) \otimes \pi'^{-1}(x)$ for any point $x \in B$. [16]

Line bundles over complex manifolds A *holomorphic line bundle* L over a complex manifold X can be defined locally by $(U_\alpha, g_{\alpha\beta})$,

$$L \longleftrightarrow (U_\alpha, g_{\alpha\beta})$$

where $\{U_\alpha\}$ is an open covering of X and $g_{\alpha\beta}$ are transition functions, i.e., $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is holomorphic functions such that $g_{\alpha\alpha} = 1$ on U_α, $g_{\alpha\beta} g_{\beta\alpha} = 1$ on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

L *and* L' are *isomorphic holomorphic line bundles* over X $\Leftrightarrow \exists$ a common open covering refinement $\{U_\alpha\}$ of X such that L *and* L' are given by $\{U_\alpha, g_{\alpha\beta}\}$ and $\{U_\alpha, g'_{\alpha\beta}\}$ respectively, and holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$g_{\alpha\beta} = f_\alpha^{-1} \cdot g'_{\alpha\beta} \cdot f_\beta, \quad \text{on } U_\alpha \cap U_\beta$$

where $f_\alpha^{-1} = \frac{1}{f_\alpha}$.

L is *trivial line bundle* if and only if the corresponding transition functions $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$ on $U_\alpha \cap U_\beta$ where $f_\alpha : U_\alpha \to \mathbb{C}^*$.

[15] cf. [H05], p. 67.
[16] cf. [H05], p. 67.

If L *and* $L^{'}$ are holomorphic line bundles given by $\{U_\alpha, g_{\alpha\beta}\}_{\alpha\in I}$ and $\{U_\alpha, g'_{\alpha\beta}\}_{\alpha\in I}$ respectively, then we define its *tensor product*, denoted as $L\otimes L'$ or $L+L'$, given by $\{U_\alpha, g_{\alpha\beta}g'_{\alpha\beta}\}_{\alpha\in I}$.

If a holomorphic line bundle L over a complex manifold X is defined by $\{U_\alpha, g_{\alpha\beta}\}_{\alpha\in I}$, then its *dual line bundle*, denoted by $-L$, or L^*, or L^{-1}, is defined by $\{U_\alpha, \frac{1}{g_{\alpha\beta}}\}$. Clearly, $L\otimes(L^{-1})$, or $L+(-L)$, is a trivial line bundle.

Holomorphic sections of a line bundle Let $\pi : L \to X$ be a holomorphic line bunlde over a complex manifold X. A *holomorphic section* of L is a holomorphic map $s : X \to L$ such that $\pi\circ s = Id$. We denote by $\Gamma(X,L)$, or $H^0(X,L)$, the set of all holomorphic sections of L over X.

Let L be given by local data $\{U_\alpha, g_{\alpha\beta}\}$.

$$\begin{array}{ccccc} L & \supset & \pi^{-1}(U_\alpha) & \xrightarrow{\theta_\alpha \simeq} & U_\alpha\times\mathbb{C}\ni (z, s_\alpha(z)) \\ & \downarrow\pi & \uparrow s & & \nearrow \\ X & \supset & U_\alpha\ni z & & \end{array}$$

On each U_α, we find a unique holomorphic function $s_\alpha\in\mathcal{O}(U_\alpha)$ so that $s(z)=\theta_\alpha^{-1}(z,s_\alpha(z))$. Then on any $U_\alpha\cap U_\beta\neq\emptyset$, we have $\theta_\alpha^{-1}(z,s_\alpha(z))=\theta_\beta^{-1}(z,s_\beta(z))$. By the linearity, we have

$$s_\alpha(z)\theta_\alpha^{-1}(z,1)=s_\beta(z)\theta_\beta^{-1}(z,1), \tag{10}$$

i.e.,

$$(z,s_\beta(z))=s_\alpha(z)\theta_\beta\circ\theta_\alpha^{-1}(z,1),$$

which implies

$$s_\beta(z)=g_{\beta\alpha}s_\alpha(z),\ \ i.e.,\ \ s_\alpha=g_{\alpha\beta}s_\beta, \qquad on\ U_\alpha\cap U_\beta.$$

Conversely, any collection $\{s_\alpha\}_{\alpha\in I}$ satisfying the above identity defines a holomorphic section $s\in\Gamma(X,L)$ by setting $s:=s_\alpha e_\alpha$, where $e_\alpha(z):=\theta_\alpha^{-1}(z,1)$.

Every bundle has a trivial section, given by $\zeta^i=0$; the graph of this section is often called the zero section. If there are no other sections, we say that *the bundle is said to have no sections*.