D'après la formule d'intégration **d'Euler-Maclaurin** on a

$$\boldsymbol{\int_a^b f(t)dt = h\frac{f(a)+f(b)}{2} + h\sum_{k=1}^{N-1} f(a+kh) - \sum_{k=1}^{\infty} B_{2k}\frac{f^{(2k-1)}(b)-f^{(2k-1)}(a)}{(2k)!}h^{2k}}$$

Donc

$$h\sum_{k=1}^{N-1} f(a+kh) = \int_a^b f(t)dt - h\frac{f(a)+f(b)}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{f^{(2k-1)}(b)-f^{(2k-1)}(a)}{(2k)!}h^{2k}$$

Posons

$$\boldsymbol{a = 0 \quad ; \quad b = x \quad ; \quad N = 2^n \quad ; \quad h = \frac{x}{2^n}}$$

$$\frac{x}{2^n}\sum_{k=1}^{2^n-1} f\left(\frac{kx}{2^n}\right) = \int_0^x f(t)dt - \frac{x}{2^n}\frac{f(0)+f(x)}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{f^{(2k-1)}(x)-f^{(2k-1)}(0)}{(2k)!}\left(\frac{x}{2^n}\right)^{2k}$$

Appliquons cette formule à la fonction **tan**

$$\frac{x}{2^n}\sum_{k=1}^{2^n-1} tan\left(\frac{kx}{2^n}\right) = \int_0^x tan(t)dt - \frac{x}{2^n}\frac{tan(0)+tan(x)}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{tan^{(2k-1)}(x)-tan^{(2k-1)}(0)}{(2k)!}\left(\frac{x}{2^n}\right)^{2k}$$

$$\frac{x}{2^n}\sum_{k=1}^{2^n-1} tan\left(\frac{kx}{2^n}\right) = -ln(\cos(x)) - \frac{x}{2^n}\frac{tan(x)}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{tan^{(2k-1)}(x)-tan^{(2k-1)}(0)}{(2k)!}\left(\frac{x}{2^n}\right)^{2k}$$

Prenons

$$x = \frac{\pi}{4}$$

$$\frac{\pi}{4\times 2^n}\sum_{k=1}^{2^n-1} tan\left(\frac{k\pi}{4\times 2^n}\right) = -ln(\cos\left(\frac{\pi}{4}\right)) - \frac{\pi}{4\times 2^n}\frac{tan\left(\frac{\pi}{4}\right)}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{tan^{(2k-1)}\left(\frac{\pi}{4}\right)-tan^{(2k-1)}(0)}{(2k)!}\left(\frac{\pi}{4\times 2^n}\right)^{2k}$$

Pour n=1

$$\frac{\pi}{8}\sum_{k=1}^{1} tan\left(\frac{k\pi}{8}\right) = -ln(\cos\left(\frac{\pi}{4}\right)) - \frac{\pi}{8}\frac{1}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{tan^{(2k-1)}\left(\frac{\pi}{4}\right)-tan^{(2k-1)}(0)}{(2k)!}\left(\frac{\pi}{8}\right)^{2k}$$

$$\sum_{k=1}^{1} tan\left(\frac{k\pi}{8}\right) = -\frac{8}{\pi}ln(\cos\left(\frac{\pi}{4}\right)) - \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{tan^{(2k-1)}\left(\frac{\pi}{4}\right)-tan^{(2k-1)}(0)}{(2k)!}\left(\frac{\pi}{8}\right)^{2k-1}$$

$$\sum_{k=1}^{\infty} B_{2k}\frac{tan^{(2k-1)}\left(\frac{\pi}{4}\right)-tan^{(2k-1)}(0)}{(2k)!}\left(\frac{\pi}{8}\right)^{2k-1} = tan\left(\frac{\pi}{8}\right) + \frac{8}{\pi}ln\left(\cos\left(\frac{\pi}{4}\right)\right) + \frac{1}{2}$$

D'après une autre étude que j'ai fait j'ai remarqué que

$$\boldsymbol{tan^{(2k-1)}\left(\frac{\pi}{4}\right) - tan^{(2k-1)}(0) = |B_{2k}|\frac{2^{2k}(2^{2k}-1)(2^{2k-1}-1)}{2k}}$$

Par un remplacement simple on aura

$$\sum_{k=1}^{\infty} B_{2k}\frac{|B_{2k}|\frac{2^{2k}(2^{2k}-1)(2^{2k-1}-1)}{2k}}{(2k)!}\left(\frac{\pi}{8}\right)^{2k-1} = tan\left(\frac{\pi}{8}\right) + \frac{8}{\pi}ln\left(\cos\left(\frac{\pi}{4}\right)\right) + \frac{1}{2}$$

$$\sum_{k=1}^{\infty} B_{2k}|B_{2k}|\frac{2^{2k}(2^{2k}-1)(2^{2k-1}-1)}{2k(2k)!}\left(\frac{\pi}{8}\right)^{2k-1} = tan\left(\frac{\pi}{8}\right) + \frac{8}{\pi}ln\left(\cos\left(\frac{\pi}{4}\right)\right) + \frac{1}{2}$$

$$2\sum_{k=1}^{\infty} B_{2k}|B_{2k}|\frac{2^{2k-1}(2^{2k}-1)(2^{2k-1}-1)}{2k(2k)!}\left(\frac{\pi}{8}\right)^{2k-1} = tan\left(\frac{\pi}{8}\right) + \frac{8}{\pi}ln\left(\cos\left(\frac{\pi}{4}\right)\right) + \frac{1}{2}$$

$$2\sum_{k=1}^{\infty} B_{2k}|B_{2k}|\frac{(2^{2k}-1)(2^{2k-1}-1)}{2k(2k)!}\left(\frac{\pi}{4}\right)^{2k-1} = tan\left(\frac{\pi}{8}\right) + \frac{8}{\pi}ln\left(\cos\left(\frac{\pi}{4}\right)\right) + \frac{1}{2}$$

Finalement

$$\boldsymbol{\sum_{k=1}^{\infty} B_{2k}|B_{2k}|\frac{(2^{2k}-1)(2^{2k-1}-1)}{2k(2k)!}x^{2k-1} = \frac{1}{2}tan\left(\frac{\pi}{8}\right) + \frac{4}{\pi}ln\left(\cos\left(\frac{\pi}{4}\right)\right) + \frac{1}{4}}$$