A conjecture about prime numbers assuming the Riemann hypothesis

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Abstract

In this paper we propose a conjecture about prime numbers. Based on the result of Pierre Dusart stating that the n^{th} prime number is smaller than $n(ln\ n + ln\ ln\ n - 0.9484)$ for $n \geq 39017$ we propose that the n^{th} prime number is smaller than $n(ln\ n + ln\ ln\ n - 1^{+})$ when $n \to +\infty$.

Keywords: Prime numbers, Dusart, Riemann hypothesis, conjecture

Conjecture. The n^{th} prime number is smaller than $n(ln\ n + ln\ ln\ n - 0.999...)$ when $n \to +\infty$

We write $ln_2\ n$ instead of $ln\ ln\ n$.

Let p(n) denote the n^{th} prime number. In this work we try to improve the result of Pierre Dusart, assuming the Riemann hypothesis and stating that for $n \geq 39017 : p(n) \leq n(ln\ n + ln\ ln\ n - 0.9484)$ [1].

Theorem. For $39017 \leq n \leq 2.10^{17}$,

$$p(n) \leq n(ln\ n + ln_2\ n - 0.9484)$$

Proof in [1]. We deduce that:

$$n(ln\ n + ln_2\ n - 0.9484) = n(ln\ n + ln_2\ n - 1 + 0.0516)$$

$$n(ln\ n + ln_2\ n - 1 + 0.0516) = n(ln\ n + ln_2\ n - 1) + 0.0516n$$

$$n(ln\ n + ln_2\ n - 1) + 0.0516n = n(ln\ n + ln_2\ n - 1) + \frac{129n}{2500}$$

If $p(n) \leq n(ln\ n + ln_2\ n - 0.9484)$ we have:

$$p(n) \leq n(ln\ n + ln_2\ n - 1) + \frac{129n}{2500}$$

.

Consequently we have

$$p(n) - (n(ln\ n + ln_2\ n - 1)) \leq n(ln\ n + ln_2\ n - 1) + \frac{129n}{2500} - (n(ln\ n + ln_2\ n - 1))\ (A)$$

According to (A) if $p(n) \leq n(ln\ n + ln_2\ n - 0.9484)$ we have:

$$\frac{n}{n(ln\ n + ln_2\ n - 1) + \frac{129n}{2500} - (n(ln\ n + ln_2\ n - 1))} \leq \frac{n}{p(n) - (n(ln\ n + ln_2\ n - 1))}$$

$$\frac{n}{\frac{129n}{2500}} \leq \frac{n}{p(n) - (n(ln\ n + ln_2\ n - 1))}$$

$$19.37984496 \leq \frac{n}{p(n) - (n(ln\ n + ln_2\ n - 1))} (B)$$

We confirmed the result (B) for $39017 \leq n \leq 2.10^{17}$ by using a statistical approach and we observe that $\frac{n}{p(n)-(n(ln\ n+ln_2\ n-1))}$ increases when n increases.

The n^{th} prime numbers were found using a list of prime numbers and with a program (see Tools).

Let x and y be two positive real numbers, we deduce:

$$p(n) \leq n(ln\ n + ln_2\ n - 1) + \frac{x.n}{y} \leftrightarrow \frac{n}{p(n) - (n(ln\ n + ln_2\ n - 1))} \geq \frac{y}{x} (C)$$

Examples. For $n = 10^5$ we have $p(10^5) = 1299709$.

$\frac{10^5}{1299709-(10^5(ln\ 10^5+ln_2\ 10^5-1))} = 24.57354013 \geq 19.37984496$

Consequently $1299709 \leq 10^5(ln\ 10^5 + ln_2\ 10^5 - 0.9484)$

In this first example we have $x = 129$ and $y = 2500$ but we can choose other values if $\frac{y}{x} \leq 24.57354013$.

For $n = 2.10^{17}$ we have $p(2.10^{17}) = 8512677386048191063$

$\frac{2.10^{17}}{8512677386048191063-(2.10^{17}(ln\ 2.10^{17}+ln_2\ 2.10^{17}-1))} = 24.099471 \geq 19.37984496$

Consequently $8512677386048191063 \leq 2.10^{17}(ln\ 2.10^{17} + ln_2\ 2.10^{17} - 0.9484)$

But if we choose $x = 2$ and $y = 48$ we have $\frac{48}{2} = 24 \leq 24.099471$ and

$$8512677386048191063 \leq 2.10^{17}(ln\ 2.10^{17} + ln_2\ 2.10^{17} - 1 + \frac{2}{48})$$

$$8512677386048191063 \leq 2.10^{17}(ln\ 2.10^{17} + ln_2\ 2.10^{17} - \frac{23}{24})$$

Conjecture. Based on our previous statistical approach we conjecture that $\frac{n}{p(n)-(n(ln\ n+ln_2\ n-1))}$ increases when $n \to +\infty$. More precisely we conjecture that $\frac{n}{p(n)-(n(ln\ n+ln_2\ n-1))} \to +\infty$ when $n \to +\infty$. For this reason we have, according to (C): $\frac{y}{x}$ that can be very high, consequently $\frac{x.n}{y} \to 0^+n$

When $n \to +\infty$ we deduce that: $p(n) \leq n(ln\, n+ln_2\, n-1)+0^+n \leftrightarrow p(n) \leq n(ln\, n+ln_2\, n-1^+)$

Finally we conjecture that, when $n \to +\infty$:

$$p(n) \leq n(ln\ n + ln_2\ n - 1^+) \leftrightarrow \frac{n}{p(n) - (n(ln\ n + ln_2\ n - 1))} \geq \frac{y}{x}$$

It remains to prove that $\frac{n}{p(n)-(n(ln\ n+ln_2\ n-1))} \to +\infty$ when $n \to +\infty$.

Tools

Statistics. Statistics were performed using Microsoft Excel 2016 and with a program.

The list of prime numbers used in this study. http://compoasso.free.fr/primelistweb/page/prime/liste_ online.php

Reference

1. PIERRE DUSART, The k^{th} prime is greater than $k(lnk + lnlnk - 1)$ for $k \geq 2$, Math. Comp. 68 (1999), 411-415