Solving Ax=b

We know that the null space of a matrix is any vector $\vec{x}$ that satisfies $A\vec{x} = \vec{O}$. But now we want the complete solution to the system represented by A. In other words, we don't just want to know which vectors $\vec{x}$ will give the zero vector as a result; now we want to know which vectors $\vec{x}$ will give *any* vector as a result.

In other words, instead of limiting ourselves to only the zero vector, now we want a way to find every $\vec{x}$ that will satisfy $A\vec{x} = \vec{b}$ when we choose any particular $\vec{b}$.

The complementary, particular, and general solutions

We can think of any $\vec{x}$ that satisfies $A\vec{x} = \vec{O}$ as part of the **complementary solution**, and any $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$ as part of the **particular solution**. The **general solution** (or the complete solution) to the system is the sum of the complementary and particular solutions.

That's because all of the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{O}$ and all the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{b}$ are part of the complete solution. To distinguish between these solution sets, we call the complementary solution $\vec{x}_n$ (since it's the null space), we call the particular solution $\vec{x}_p$, and we call the general solution just $\vec{x}$.

Then we can say $A\vec{x}_n = \vec{O}$ and $A\vec{x}_p = \vec{b}$. If we add these together, we get

$$A\vec{x}_n + A\vec{x}_p = \vec{O} + \vec{b}$$

$$A(\vec{x}_n + \vec{x}_p) = \vec{b}$$

What this equation shows us is that the full solution set of vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{b}$ will be any vector $\vec{x} = \vec{x}_n + \vec{x}_p$. And that's how we can conclude that the complete solution will be the sum of the complementary and particular solutions.

If you've taken a Differential Equations course (it's okay if you haven't), this should remind you of solving non-homogeneous differential equations. In both cases (here in Linear Algebra with matrices, and in Differential Equations with non-homogeneous equations), we find the set of solutions that satisfy the homogeneous equation where the right side is 0 (or the zero vector $\vec{O}$), and then we find the particular solution that satisfies the non-zero right side. Then the general solution is the sum of the two.

Finding the complete solution set

So to find the full family of solutions to $A\vec{x} = \vec{b}$, we first find the set of all the linear combinations of column vectors that are solutions to the null space equation.

Then we find the particular solution by setting all the free variables equal to 0 (if there *is* any solution, we can find it by setting the free variables equal to 0), plugging in a $\vec{b}$ that satisfies any constraint on $\vec{b}$, and then plugging in to get the single particular vector.

Then the general solution is the sum of the particular solution and the complementary solution.

Let's work through a full example so that we can see how to get all the way to $\vec{x}=\vec{x}_n+\vec{x}_p$.

Example

Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}1 & 2 & 3 & 3\\ 2 & 2 & 1 & 4\\ 3 & 4 & 4 & 7\end{bmatrix}$$

To find the general solution to $A\vec{x}=\vec{b}$, we need to find the solutions in the null space, and then the particular solution. Let's start with the solutions in the null space, which we'll find by solving $A\vec{x}=\vec{O}$. To get those null space solutions, we'll augment the matrix.

$$\begin{bmatrix}1 & 2 & 3 & 3 & | & 0\\ 2 & 2 & 1 & 4 & | & 0\\ 3 & 4 & 4 & 7 & | & 0\end{bmatrix}$$

We want to put the augmented matrix into reduced row-echelon form. The first column already contains a pivot of 1, so we'll zero out the rest of the first column.

$$\begin{bmatrix}1 & 2 & 3 & 3 & | & 0\\ 0 & -2 & -5 & -2 & | & 0\\ 3 & 4 & 4 & 7 & | & 0\end{bmatrix}\rightarrow\begin{bmatrix}1 & 2 & 3 & 3 & | & 0\\ 0 & -2 & -5 & -2 & | & 0\\ 0 & -2 & -5 & -2 & | & 0\end{bmatrix}$$

The next step would be to multiply through the second row to find a pivot of 1. But looking ahead, we can see that the second and third rows are identical, so let's use $R_3 - R_2 \to R_3$ to make our calculations easier.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & 0 \\ 0 & -2 & -5 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Now find the pivot of 1 in the second row.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & 0 \\ 0 & 1 & \frac{5}{2} & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

There can't be a pivot in the third column, so we'll move to the fourth column. The fourth column can't have a pivot either. In fact, the row of zeros at the bottom of the matrix tells us that the last row is a multiple of some other row or rows in the matrix. In fact, in the original matrix A, we can see that the third row is the sum of the first and second rows.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & 0 \\ 0 & 1 & \frac{5}{2} & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

With the matrix in reduced row-echelon form, we can see that the first and second columns are pivot columns and the third and fourth columns are free columns. Which means x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables. Let's parse out a system of equations.

$$1x_1 + 0x_2 - 2x_3 + 1x_4 = 0$$

$$0x_1 + 1x_2 + \frac{5}{2}x_3 + 1x_4 = 0$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

The system simplifies to

$$x_1 - 2x_3 + x_4 = 0$$

$$x_2 + \frac{5}{2}x_3 + x_4 = 0$$

Solve for the pivot variables in terms of the free variables.

$$x_1 = 2x_3 - x_4$$

$$x_2 = -\frac{5}{2}x_3 - x_4$$

Then the vectors that satisfy the null space are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

In other words, any linear combination of these column vectors is a member of the null space; it satisfies $A\vec{x}_n = \vec{O}$. We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. So instead of augmenting the matrix with the zero vector, we augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 2 & 2 & 1 & 4 & | & b_2 \\ 3 & 4 & 4 & 7 & | & b_3 \end{bmatrix}$$

Now we'll again put the matrix into reduced row-echelon form. Zero out the first column below the pivot.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 0 & -2 & -5 & -2 & | & b_2 - 2b_1 \\ 3 & 4 & 4 & 7 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 0 & -2 & -5 & -2 & | & b_2 - 2b_1 \\ 0 & -2 & -5 & -2 & | & b_3 - 3b_1 \end{bmatrix}$$

Find the pivot in the second column.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}b_2 + b_1 \\ 0 & -2 & -5 & -2 & | & b_3 - 3b_1 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & -b_1+b_2 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}b_2+b_1 \\ 0 & -2 & -5 & -2 & | & b_3-3b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & | & -b_1+b_2 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}b_2+b_1 \\ 0 & 0 & 0 & 0 & | & -b_1-b_2+b_3 \end{bmatrix}$$

With the matrix in reduced row-echelon form, we can see that we have a constraint on $\vec{b}$. From the bottom row of the rref matrix, we get

$$0 = -b_1 - b_2 + b_3$$

We need to pick a set of values $\vec{b} = (b_1, b_2, b_3)$ that will satisfy this equation. It doesn't matter which values we choose, as long as they make the equation true. Let's choose $b_1 = 1$, $b_2 = 2$, and $b_3 = 3$.

$$0 = -b_1 - b_2 + b_3$$

$$0 = -1 - 2 + 3$$

$$0 = 0$$

Then we could rewrite the rref matrix as

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & -1+2 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}(2)+1 \\ 0 & 0 & 0 & 0 & | & -1-2+3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & | & 1 \\ 0 & 1 & \frac{5}{2} & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables. To find the vectors that satisfy $A\vec{x}_p = \vec{b}$, we need to set the free variables equal to 0. So let's first rewrite the matrix as a system of equations.

$$1x_1 + 0x_2 - 2x_3 + 1x_4 = 1$$

$$0x_1 + 1x_2 + \frac{5}{2}x_3 + 1x_4 = 0$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

The system simplifies to

$$x_1 - 2x_3 + x_4 = 1$$

$$x_2 + \frac{5}{2}x_3 + x_4 = 0$$

Now set the free variables equal to 0, $x_3 = 0$ and $x_4 = 0$.

$$x_1 - 2(0) + 0 = 1$$

$$x_2 + \frac{5}{2}(0) + 0 = 0$$

The system becomes

$$x_1 = 1$$

$$x_2 = 0$$

So the particular solution then is $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This is the complete solution to the system that's represented by matrix A.