SOLVING AX=B

1. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}2 & -4 & 3 & -6\\ 1 & -2 & 0 & 0\\ 4 & -8 & 5 & -10\end{bmatrix}$$

Solution:

Solve $A\vec{x}=\vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & -4 & 3 & -6 & | & 0\\ 1 & -2 & 0 & 0 & | & 0\\ 4 & -8 & 5 & -10 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 2 & -4 & 3 & -6 & | & 0\\ 4 & -8 & 5 & -10 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 3 & -6 & | & 0\\ 4 & -8 & 5 & -10 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 3 & -6 & | & 0\\ 0 & 0 & 5 & -10 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 1 & -2 & | & 0\\ 0 & 0 & 5 & -10 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 1 & -2 & | & 0\\ 0 & 0 & 0 & 0 & | & 0\end{bmatrix}$$

The first and third columns are pivot columns and the second and fourth columns are free columns. Which means x_1 and x_3 are pivot variables, and x_2 and x_4 are free variables. Pull out a system of equations,

$$x_1-2x_2=0$$

$$x_3-2x_4=0$$

then solve it for the pivot variables in terms of the free variables.

$$x_1 = 2x_2$$

$$x_3 = 2x_4$$

Express the system as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2, b_3)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & -4 & 3 & -6 & | & b_1 \\ 1 & -2 & 0 & 0 & | & b_2 \\ 4 & -8 & 5 & -10 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 2 & -4 & 3 & -6 & | & b_1 \\ 4 & -8 & 5 & -10 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 3 & -6 & | & b_1 - 2b_2 \\ 4 & -8 & 5 & -10 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 3 & -6 & | & b_1 - 2b_2 \\ 0 & 0 & 5 & -10 & | & b_3 - 4b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}b_1 - \frac{2}{3}b_2 \\ 0 & 0 & 5 & -10 & | & b_3 - 4b_2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}b_1 - \frac{2}{3}b_2 \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3}b_1 - \frac{2}{3}b_2 + b_3 \end{bmatrix}$$

From the bottom row of rref(A), we get

$$0 = -\frac{5}{3}b_1 - \frac{2}{3}b_2 + b_3$$

$$0 = -5b_1 - 2b_2 + 3b_3$$

Pick a set of values $\overrightarrow{b} = (b_1, b_2, b_3)$ that satisfies this equation. The vector $\overrightarrow{b} = (1,2,3)$ will work.

$$0 = -5(1) - 2(2) + 3(3)$$

$$0 = -5 - 4 + 9$$

$$0 = 0$$

Rewrite rref(A) with $\overrightarrow{b} = (1,2,-3)$.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}(1) - \frac{2}{3}(2) \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3}(1) - \frac{2}{3}(2) + 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3} - \frac{4}{3} \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3} - \frac{4}{3} + \frac{9}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & -\frac{3}{3} \\ 0 & 0 & 0 & 0 & | & \frac{0}{3} \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_3 are pivot variables, and x_2 and x_4 are free variables. Pull a system of equations from the matrix,

$$x_1 - 2x_2 = 2$$

$$x_3 - 2x_4 = -1$$

then set the free variables equal to 0 to simplify the system to

$$x_1 - 2(0) = 2$$

$$x_3 - 2(0) = -1$$

and then

$$x_1 = 2$$

$$x_3 = -1$$

So the particular solution then is $x_1 = 2$, $x_2 = 0$, $x_3 = -1$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

■ 2. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 3 & 6 \\ 6 & 12 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Solution:

Solve $A\vec{x} = \vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 3 & 6 & | & 0 \\ 6 & 12 & | & 0 \\ 1 & 1 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 6 & 12 & | & 0 \\ 1 & 1 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \\ 1 & 1 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \\ 0 & -1 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \\ 0 & -1 & | & 0 \\ 0 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & -1 & | & 0 \\ 0 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$