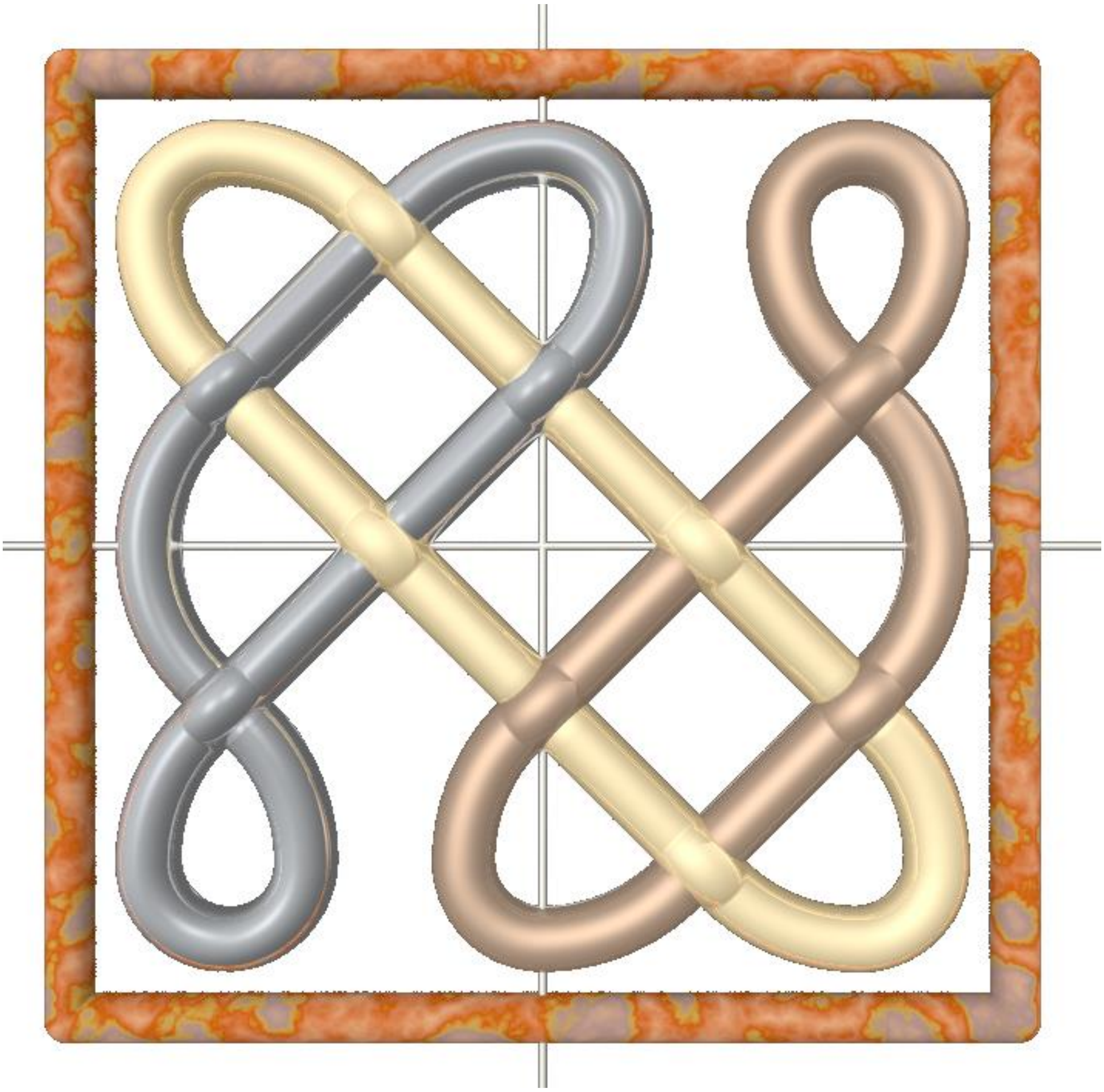

How many Celtic Knots ?

Médiat

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Réalisé en LaTeX

1 Introduction

1.1 What is a Celtic Knot

Celtic knots are interlaced patterns, made of ropes or threads tied together. They appear in the art of various peoples including Celts (obviously), Romans, Vikings and Saxons

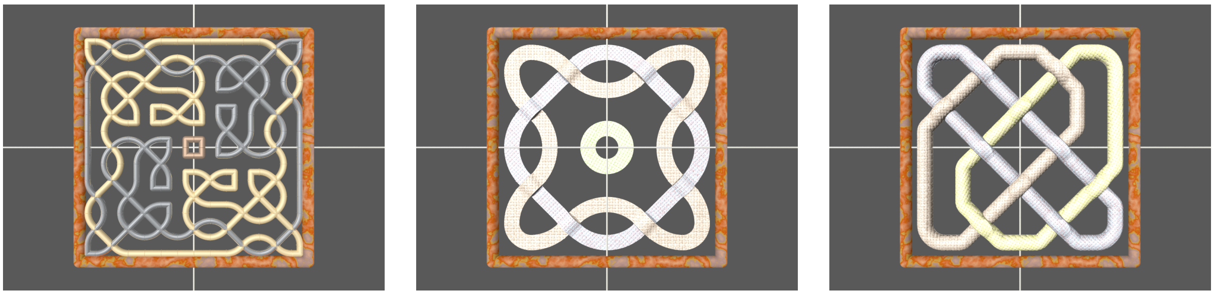

FIGURE 1 – *Some examples*

1.2 References

It is easy to find sites about Celtic Knots on the net, here are some examples :

1. *Discover the Meaning Behind These Intricate Designs*.
2. *History and Symbolism* .
3. *A Brief History of Celtic Knots* .

2 How to build a Celtic Knot

Please note that we use the same symbol $K_{n,m}$ to designate the set of Celtic Knot of size $n \times m$ (or $m \times n$) and the cardinal of this set, but there should be no confusion.

2.1 Algorithm

In this section we describe the "usual" algorithm (in fact only the first part which is the only one needed for our goal), and then we'll make some changes of notation to ease some calculations.

To build a Celtic Knot of size $n \times m$, we first draw the boundaries of a $2n \times 2m$ rectangle using coordinates : $[\![0, 2n]\!] \times [\![0, 2m]\!]$

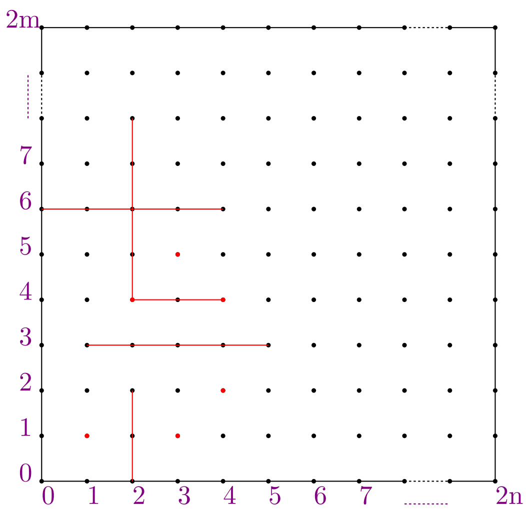

FIGURE 2 – *Creating a Celtic Knot*

A point inside the rectangle is said "even" if the sum of its coordinates is even, and, of course, is said "odd" if the sum of its coordinates is odd. For instance, the red dots in the previous pictures are examples of even points.

After the boundaries, we add barriers which are segment, horizontal, or vertical, starting on an even point, ending on an even point, and crossing other barriers on even points.

The red segments in the previous figure are examples of valid barriers.

Once the Boundaries and Barriers are set, there is an additionnal work to do to draw an actual Celtic Knot, but this part is not relevant to our purpose, and can be found on several sites, see the subsection 2.3.

2.2 Preparing the Calculations

As said before we will make a slight change : we no longer use the system of coordinates $[\![0, 2n]\!] \times [\![0, 2m]\!]$ but a new one translated from the previous : $[\![-n, n]\!] \times [\![-m, m]\!]$, therefore we need to make a small modification in a definition : a point of coordinates (i, j) in the new system is said "even" if $(i + n) + (j + m)$ is even, and is said "odd" if $(i + n) + (j + m)$ is odd.

The odd points, of coordinates (i, j) that are not on a boundary, are the center of squares such as the horizontal segment $(i - 1, j) - (i + 1, j)$ and the vertical segment $(i, j - 1) - (i, j + 1)$ are valid barriers, but we cannot have both (they would cross on an odd point), and all the barriers can be build this way, we will call these squares "Generating Squares".

2.3 Reference :

1. *MathRecreation*.
2. *Celtic Knot Theory* .
3. *A Celtic Framework for Knots and Links* .

3 How to count the number of (n, m) Celtic Knots

3.1 $K_{n,1}$ a simple example

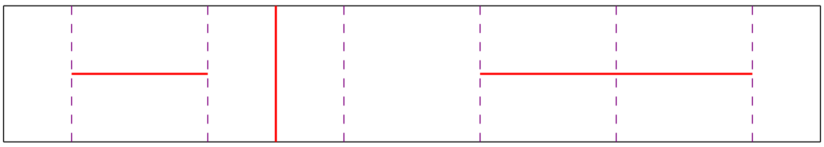

FIGURE 3 – *n × 1 Celtic Knot*

The first and last half square are (and must be) empty, but the squares delimited by the violet dashed lines, can contain, a vertical line, a horizontal line, or neither but not both (or anything else), those are the Generating Squares.

We can easily "code" such a design :

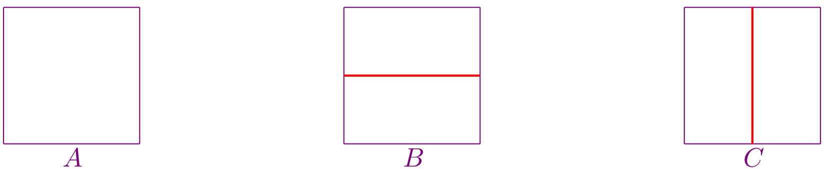

FIGURE 4 – *Coding of Barriers*

To count $K_{(n,1)}$, all we have to do is to count the number of words of length $(n - 1)$ we can build with an alphabet of three different letters. This is easily done with two recursions (one for even numbers and one for odd numbers), or you can have a look on Sequence A032120 on the OEIS web site.

The result can be written as :

$$\begin{cases} K_{1,2n} & = \dfrac{3^{2n-1}+3^n}{2} \\ K_{1,2n+1} & = \dfrac{3^{2n}+3^n}{2} \end{cases}$$

3.2 $K_{2,2}$ another simple example

This case is simple enough to be counted by hand :

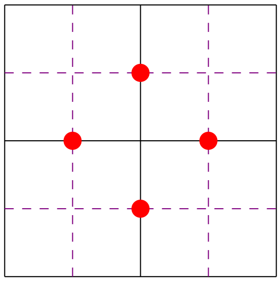

FIGURE 5 – *2 × 2 Celtic Knot*

The red dots are placed at the center of the four possible Generating Squares.

The Generating Squares are one of the 3 types described in Figure 4 we can list the differents cases :

1. The four are identical we note this case [4]
2. Three are identical and the fourth is different, we note this case [3, 1]
3. Two are identical and the two others are identical to each other but different of the two first ones, we note this case [2, 2]
4. Two are identical and the two others are different of each other and different of the two first ones, we note this case [2, 1, 1]

It is easily seen that this description gives all the cases and that they are disjoint.

Case	Explanation	Calculation	Result
[4]	We must choose one of the Three type of Squares and put it in the four places	$\binom{3}{1}$	3
[3, 1]	We must choose one of the Three type of Squares and put it anywhere, then we choose another type among the two remaining ones and place it at the three remaining slots	$\binom{3}{1} \times \binom{2}{1}$	6
[2, 2]	We must choose two of the Three type of Squares and place them such as two identical squares are either side by side or on the same diagonal	$\binom{3}{2} \times 2$	6
[2, 1, 1]	We must choose one of the Three type of Squares and put the two identical squares either side by side or on the same diagonal, the two remaining types being placed in the remaining slots	$\binom{3}{1} \times 2$	6
		$K_{2,2}$	21

4 Description

4.1 Let's picture it !

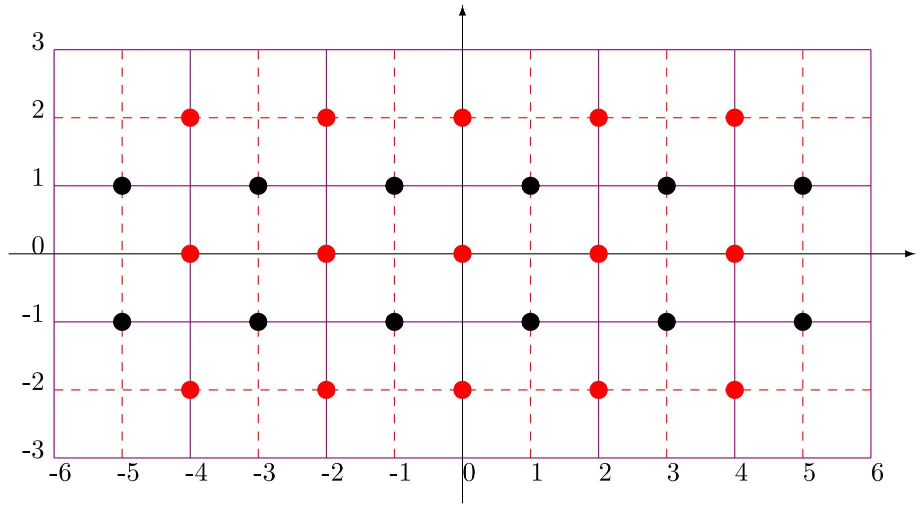

FIGURE 6 – *6 × 3 Celtic Knot*

The dots in the picture are the centers of the Generating Squares.

4.2 Coordinates of the red dots

$$(2i + (n \bmod 2), 2j + 1 - (m \bmod 2))$$

with the following constraints :

$$\begin{cases} -n+2 & \leq & 2i + (n \bmod 2) & \leq & n-2 \\ -m+1 & \leq & 2j + 1 - (m \bmod 2) & \leq & m-1 \end{cases}$$

We can rewrite the first inequation as :

$$-n + 2 - (n \bmod 2) \leq 2i \leq n - 2 - (n \bmod 2)$$

It is easily seen that $-n + 2 - (n \bmod 2)$ and $n - 2 - (n \bmod 2)$ are even integers, so we can rewrite the former inequation as

$$\frac{-n + 2 - (n \bmod 2)}{2} \leq i \leq \frac{n - 2 - (n \bmod 2)}{2}$$

where the limits are integers, so the number of valid values for i is

$$\frac{n - 2 - (n \bmod 2)}{2} - \frac{-n + 2 - (n \bmod 2)}{2} + 1 = n - 1$$

We can rewrite the second inequation as :

$$-m + (m \bmod 2) \leq 2j \leq m - 2 + (m \bmod 2)$$

It is easily seen that $-m + (m \bmod 2)$ and $m - 2 + (m \bmod 2)$ are even integers, so we can rewrite the former inequation as

$$\frac{-m + (m \bmod 2)}{2} \leq j \leq \frac{m - 2 + (m \bmod 2)}{2}$$

where the limits are integers, so the number of valid values for j is

$$\frac{m - 2 + (m \bmod 2)}{2} - \frac{-m + (m \bmod 2)}{2} + 1 = m$$

Hence the number of red dots is $m(n - 1)$.

4.3 Coordinates of the black dots

$$(2i + 1 - (n \bmod 2), 2j + (m \bmod 2))$$

with the following constraints :

$$\begin{cases} -n+1 & \leq & 2i+1-(n \bmod 2) & \leq & n-1 \\ -m+2 & \leq & 2j+(m \bmod 2) & \leq & m-2 \end{cases}$$

We can rewrite the first inequation as :

$$-n + (n \bmod 2) \leq 2i \leq n - 2 + (n \bmod 2)$$

It is easily seen that $-n+(n \bmod 2)$ and $n-2+(n \bmod 2)$ are even integers, so we can rewrite the former inequation as

$$\frac{-n + (n \bmod 2)}{2} \leq i \leq \frac{n - 2 + (n \bmod 2)}{2}$$

where the limits are integers, so the number of valid values for i is

$$\frac{n - 2 + (n \bmod 2)}{2} - \frac{-n + (n \bmod 2)}{2} + 1 = n$$

We can rewrite the second inequation as :

$$-m + 2 - (m \bmod 2) \leq 2j \leq m - 2 - (m \bmod 2)$$

It is easily seen that $-m+2-(m \bmod 2)$ and $m-2-(m \bmod 2)$ are even integers, so we can rewrite the former inequation as

$$\frac{-m + 2 - (m \bmod 2)}{2} \leq j \leq \frac{m - 2 - (m \bmod 2)}{2}$$

where the limits are integers, so the number of valid values for j is

$$\frac{m - 2 - (m \bmod 2)}{2} - \frac{-m + 2 - (m \bmod 2)}{2} + 1 = m - 1$$

Hence the number of red dots is $n(m-1)$.

The total number of Generating Squares is $2nm - (n+m)$

5 Symmetry Group

5.1 Burnside's lemma

Burnside's lemma, (which is also known under several other names as the orbit-counting theorem) is a result in group theory which is often useful in taking account of symmetry when counting mathematical objects.

Let G be a finite group that acts on a set X. For each $g \in G$ let X^g denote the set of elements in X that are fixed by g (also said to be left invariant by g), i.e. $X^g = \{x \in X | g.x = x\}$. Burnside's lemma asserts the following formula for the number of orbits, denoted $|X/G|$:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

In the case of the Celtic Knot, the number of orbits for all the elements of G is exactly the number of configurations that cannot be obtained from another one (not in the same orbit) using the elements of G.

Thanks to Brian Hopkins (on stackoverflow) who led me to the ideas used in this document and thanks to William Burnside fot this magic lemma.

5.2 Application Celtic Knots

For a square Celtic Knot the symmetry group is $\{Id, R_{90}, R_{180}, R_{270}, S_H, S_V, S_{D_1}, S_{D_2}\}$, for a non square Celtic Knot, the group reduces to $\{Id, R_{180}, S_H, S_V\}$ exept for the $n \times 1$ Celtic Knot for which it reduces further to $\{Id, S_V\}$ ($S_H = Id$ et $R_{180} = S_V$)

5.2.1 Identity

Of course this case is very easy, as all elements ares fixed by the Identity, so the number of fixed points by the Identity is $3^{2mn-(m+n)}$ which, in case of a square Celtic Knot is equal to $3^{2n(n-1)}$.

5.2.2 Rotation 90°

Of course this apply only to square Celtic Knots. To define a fixed point for R_{90} we can first notice that the center of the square is an even point ($0+n+0+n=2n$ is even), so is not a Generating Square, and then we have to choose the content (3 possibilities) of the Generating Squares lying in one quadrant, let say $x>0$ and $y\geq 0$, (the fourth of the total) and force the other Generating Square's content according to the rule examplified in Figure 7.

Therefore the number of fixed points for R_{90} is $3^{\frac{n(n-1)}{2}}$

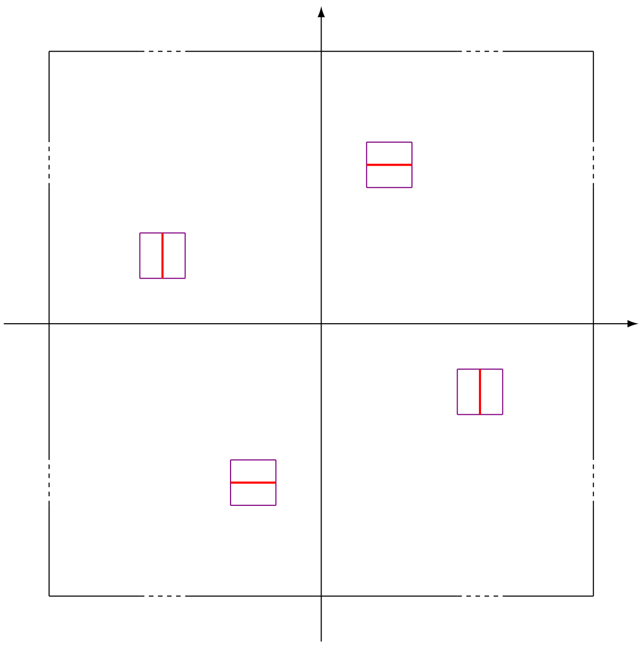

FIGURE 7 – *Rotation 90°*

5.2.3 Rotation 180°

If the center is not a Generating Square, which happen when $(0+n+0+m)$is even, we have to choose the content (3 possibilities) of half the Generating Squares and force the other Generating Square's content according to the rule : the content of square (i,j) must be equal to the content of the square $(-i,-j)$, that is to say we have to choose the content (3 posibilities) of half the Generating Squares. In this case the number of fixed points is $3^{\frac{2mn-(n+m)}{2}}$ which, in case of a square Celtic Knot gives $3^{n(n-1)}$

If the center is a Generating Square, which happen when $(0+n+0+m)$is odd, we have to choose the content of this one square and then the content (3 possibilities) of half the other Generating Squares and force the other Generating Square's content according to the same rule as in the previous case. In this case the number of fixed points is $3^{\frac{2mn-(n+m)+1}{2}}$. This does not happen with Square Celtic Knot.

5.2.4 Rotation 270°

Exactely like sub-subsection 5.2.2

Therefore the number of fixed points with R_{270} is $3^{\frac{n(n-1)}{2}}$

5.2.5 Horizontal Symmetry

To define a fixed point for S_H we have first to count the Generating Squares lying on the line defined by $j=0$, and it depends if m is even or odd.

If m is even there are n Generating Squares on the line $j = 0$, all of them being fixed, which means that to define a fixed point we must choose the content of these n ones and of half the rest that is $n + \frac{2mn-(n+m)-n}{2} = \frac{2mn-m}{2}$ leading to $3^{\frac{2mn-m}{2}}$ fixed points.

If m is odd there are $n-1$ Generating Squares on the line $j = 0$, all of them being fixed, which means that to define a fixed point we must choose the content of these $n-1$ ones and of half the rest that is $n - 1 + \frac{2mn-(n+m)-n+1}{2} = \frac{2mn-m-1}{2}$ leading to $3^{\frac{2mn-m+1}{2}}$ fixed points.

Therefore the number of fixed points by S_H is :

$$\begin{cases} n \text{ is even} & \rightarrow \quad 3^{\frac{2nm-m}{2}} \\ n \text{ is odd} & \rightarrow \quad 3^{\frac{2nm-m-1}{2}} \end{cases}$$

And in case of a square Celtic Knot $(m = n)$:

$$\begin{cases} n \text{ is even} & \rightarrow \quad 3^{\frac{2n^2-n}{2}} \\ n \text{ is odd} & \rightarrow \quad 3^{\frac{2n^2-n-1}{2}} \end{cases}$$

5.2.6 Vertical Symmetry

This case is the same as sub-subsection 5.2.5, if we switch the roles of m and n.
Therefore the number of fixed points by S_V is :

$$\begin{cases} n \text{ is even} & \rightarrow \quad 3^{\frac{2nm-n}{2}} \\ n \text{ is odd} & \rightarrow \quad 3^{\frac{2nm-n-1}{2}} \end{cases}$$

And in case of a square Celtic Knot $(m = n)$:

$$\begin{cases} n \text{ is even} & \rightarrow \quad 3^{\frac{2n^2-n}{2}} \\ n \text{ is odd} & \rightarrow \quad 3^{\frac{2n^2-n-1}{2}} \end{cases}$$

5.2.7 First Diagonal Symmetry

Of course this apply only to square Celtic Knots. To define a fixed point for S_{D_1} we can first notice that none of the squares lying on the first diagonal (whose coordinates are (i, i)) are Generating, because $i+n+i+n$ is even. So, to build a fixed point we have to choose the content of half of the squares and force the other Generating Square's content according to the rule : the content of square (i, j) must be equal to the content of the square (j, i), therefore the number of fixed points with S_{D_1} is $3^{n(n-1)}$

5.2.8 Second Diagonal Symmetry

Exactely like sub-subsection 5.2.7
Therefore the number of fixed points with S_{D_2} is $3^{n(n-1)}$.

5.3 Examples

For $K_{(1,n)}$, there are $(n-1)$ Generating Square, so the Identity possesses 3^{n-1} fixed point, for the reflection S_V we have to consider two differents cases :

1. $n = 2k$ (n is even), there are $2k-1$ Generating Squares, one in the middle which is its own image and the left half of the others are images of the right half, so the number of fixed points (for the empty squares) is $1 + \frac{2k-1-1}{2} = k = \frac{n}{2}$, which leads to 3^k fixed points.
2. $n = 2k + 1$ (n is odd), there are $2k$ Generating Squares, the left half are images of the right half, so the number of fixed points (for the empty squares) is $k = \frac{n-1}{2}$, which leads to 3^k fixed points.

Which are the results of subsection 3.1

For $K_{2,2}$ we get $K_{2,2} = \dfrac{3^4 + 3^1 + 3^2 + 3^1 + 3^3 + 3^3 + 3^2 + 3^2}{8} = 21$

Which are the results of subsection 3.2

6 Last Remark

All the previous calculations can be checked using the following table and the coordinates calculated in subsection 4.2 and subsection 4.3.

Operation	$(\ x,\ \ y)$
Id	$(\ x,\ \ y)$
R_{90}	$(-y,\ \ x)$
R_{180}	$(-x, -y)$
R_{270}	$(\ y, -x)$
S_H	$(\ x, -y)$
S_V	$(-x,\ \ y)$
S_{D_1}	$(\ y,\ \ x)$
S_{D_2}	$(-y, -x)$

7 Summary

7.1 Non Square Celtic Knot

Operation	n even / m even	n even / m odd	n odd / m even	n odd / m odd
Id	$3^{2nm-(n+m)}$	$3^{2nm-(n+m)}$	$3^{2nm-(n+m)}$	$3^{2nm-(n+m)}$
R_{180}	$3^{\frac{2nm-(n+m)}{2}}$	$3^{\frac{2nm-(n+m)+1}{2}}$	$3^{\frac{2nm-(n+m)+1}{2}}$	$3^{\frac{2nm-(n+m)}{2}}$
S_H	$3^{\frac{2nm-m}{2}}$	$3^{\frac{2nm-m-1}{2}}$	$3^{\frac{2nm-m}{2}}$	$3^{\frac{2nm-m-1}{2}}$
S_V	$3^{\frac{2nm-n}{2}}$	$3^{\frac{2nm-n}{2}}$	$3^{\frac{2nm-n-1}{2}}$	$3^{\frac{2nm-n-1}{2}}$

n even / m even	$K_{(n,m)} =$	$\dfrac{3^{2nm-(n+m)} + 3^{\frac{2nm-(n+m)}{2}} + 3^{\frac{2nm-m}{2}} + 3^{\frac{2nm-n}{2}}}{4}$
n even / m odd	$K_{(n,m)} =$	$\dfrac{3^{2nm-(n+m)} + 3^{\frac{2nm-(n+m)+1}{2}} + 3^{\frac{2nm-m-1}{2}} + 3^{\frac{2nm-n}{2}}}{4}$
n odd / m even	$K_{(n,m)} =$	$\dfrac{3^{2nm-(n+m)} + 3^{\frac{2nm-(n+m)+1}{2}} + 3^{\frac{2nm-m}{2}} + 3^{\frac{2nm-n-1}{2}}}{4}$
n odd / m odd	$K_{(n,m)} =$	$\dfrac{3^{2nm-(n+m)} + 3^{\frac{2nm-(n+m)}{2}} + 3^{\frac{2nm-m-1}{2}} + 3^{\frac{2nm-n-1}{2}}}{4}$

7.2 Square Celtic Knot

Operation	n even	n odd
Id	3^{2n^2-2n}	3^{2n^2-2n}
R_{90}	$3^{\frac{n(n-1}{2}}$	$3^{\frac{n(n-1}{2}}$
R_{180}	$3^{n(n-1)}$	$3^{n(n-1)}$
R_{270}	$3^{\frac{n(n-1}{2}}$	$3^{\frac{n(n-1}{2}}$
S_H	$3^{\frac{2n^2-n}{2}}$	$3^{\frac{2n^2-n-1}{2}}$
S_V	$3^{\frac{2n^2-n}{2}}$	$3^{\frac{2n^2-n-1}{2}}$
S_{D_1}	$3^{n(n-1)}$	$3^{n(n-1)}$
S_{D_2}	$3^{n(n-1)}$	$3^{n(n-1)}$

n even	$K_{(n,n)} =$	$K_{(n,n)} = \frac{3^{2n^2-2n} + 2.3^{\frac{n(n-1}{2}} + 3.3^{n(n-1)} + 2.3^{\frac{2n^2-n}{2}}}{8}$
n odd	$K_{(n,n)} =$	$K_{(n,n)} = \frac{3^{2n^2-2n} + 2.3^{\frac{n(n-1}{2}} + 3.3^{n(n-1)} + 2.3^{\frac{2n^2-n-1}{2}}}{8}$