If $z^n=u_n+iv_n$, then $z^{n+1}=u_{n+1}+iv_{n+1}$ where

**3.7.23** $u_{n+1}=xu_n-yv_n;\ v_{n+1}=xv_n+yu_n$

$\mathscr{R}z^n$ and $\mathscr{I}z^n$ are called harmonic polynomials.

**3.7.24** $$\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{x-iy}{x^2+y^2}$$

**3.7.25** $$\frac{1}{z^n}=\frac{\bar{z}^n}{|z|^{2n}}=(z^{-1})^n$$

### Roots

**3.7.26** $z^{\frac{1}{2}}=\sqrt{z}=r^{\frac{1}{2}}e^{\frac{1}{2}i\theta}=r^{\frac{1}{2}}\cos\frac{1}{2}\theta+ir^{\frac{1}{2}}\sin\frac{1}{2}\theta$

If $-\pi<\theta\leq\pi$ this is the principal root. The other root has the opposite sign. The principal root is given by

**3.7.27** $z^{\frac{1}{2}}=[\frac{1}{2}(r+x)]^{\frac{1}{2}}\pm i[\frac{1}{2}(r-x)]^{\frac{1}{2}}=u\pm iv$ where $2uv=y$ and where the ambiguous sign is taken to be the same as the sign of $y$.

**3.7.28** $z^{1/n}=r^{1/n}e^{i\theta/n}$, (principal root if $-\pi<\theta\leq\pi$). Other roots are $r^{1/n}e^{i(\theta+2\pi k)/n}$ $(k=1, 2, 3, \ldots, n-1)$.

### Inequalities

**3.7.29** $$\left||z_1|-|z_2|\right|\leq|z_1\pm z_2|\leq|z_1|+|z_2|$$

### Complex Functions, Cauchy-Riemann Equations

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ where $u(x,y)$, $v(x,y)$ are real, is *analytic* at those points $z=x+iy$ at which

**3.7.30** $$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

If $z=re^{i\theta}$,

**3.7.31** $$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}$$

### Laplace's Equation

The functions $u(x,y)$ and $v(x,y)$ are called harmonic functions and satisfy Laplace's equation:

#### Cartesian Coordinates

**3.7.32** $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

#### Polar Coordinates

**3.7.33** $$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial \theta^2}=r\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial \theta^2}=0$$

## 3.8. Algebraic Equations

### Solution of Quadratic Equations

**3.8.1** Given $az^2+bz+c=0$,

$$z_{1,2}=-\left(\frac{b}{2a}\right)\pm\frac{1}{2a}q^{\frac{1}{2}},\ q=b^2-4ac,$$

$$z_1+z_2=-b/a,\ z_1z_2=c/a$$

If $q>0$, two real roots,
$q=0$, two equal roots,
$q<0$, pair of complex conjugate roots.

### Solution of Cubic Equations

**3.8.2** Given $z^3+a_2z^2+a_1z+a_0=0$, let

$$q=\frac{1}{3}a_1-\frac{1}{9}a_2^2;\ r=\frac{1}{6}(a_1a_2-3a_0)-\frac{1}{27}a_2^3.$$

If $q^3+r^2>0$, one real root and a pair of complex conjugate roots,
$q^3+r^2=0$, all roots real and at least two are equal,
$q^3+r^2<0$, all roots real (irreducible case).

Let

$$s_1=[r+(q^3+r^2)^{\frac{1}{2}}]^{\frac{1}{3}},\ s_2=[r-(q^3+r^2)^{\frac{1}{2}}]^{\frac{1}{3}}$$

then

$$z_1=(s_1+s_2)-\frac{a_2}{3}$$

$$z_2=-\frac{1}{2}(s_1+s_2)-\frac{a_2}{3}+\frac{i\sqrt{3}}{2}(s_1-s_2)$$

$$z_3=-\frac{1}{2}(s_1+s_2)-\frac{a_2}{3}-\frac{i\sqrt{3}}{2}(s_1-s_2).$$

If $z_1$, $z_2$, $z_3$ are the roots of the cubic equation

$$z_1+z_2+z_3=-a_2$$

$$z_1z_2+z_1z_3+z_2z_3=a_1$$

$$z_1z_2z_3=-a_0$$

### Solution of Quartic Equations

**3.8.3** Given $z^4+a_3z^3+a_2z^2+a_1z+a_0=0$, find the real root $u_1$ of the cubic equation

$$u^3-a_2u^2+(a_1a_3-4a_0)u-(a_1^2+a_0a_3^2-4a_0a_2)=0$$

and determine the four roots of the quartic as solutions of the two quadratic equations

$$v^2+\left[\frac{a_3}{2}\mp\left(\frac{a_3^2}{4}+u_1-a_2\right)^{\frac{1}{2}}\right]v+\frac{u_1}{2}\mp\left[\left(\frac{u_1}{2}\right)^2-a_0\right]^{\frac{1}{2}}=0$$

If all roots of the cubic equation are real, use the value of $u_1$ which gives real coefficients in the *quadratic equation and select signs so that if

$$z^4+a_3z^3+a_2z^2+a_1z+a_0=(z^2+p_1z+q_1)(z^2+p_2z+q_2),$$

then

$$p_1+p_2=a_3,\ p_1p_2+q_1+q_2=a_2,\ p_1q_2+p_2q_1=a_1,\ q_1q_2=a_0.$$

If $z_1$, $z_2$, $z_3$, $z_4$ are the roots,

$$\Sigma z_i=-a_3,\ \Sigma z_iz_jz_k=-a_1,$$

$$\Sigma z_iz_j=a_2,\ z_1z_2z_3z_4=a_0.$$

## 3.9. Successive Approximation Methods

### General Comments

**3.9.1** Let $x=x_1$ be an approximation to $x=\xi$ where $f(\xi)=0$ and both $x_1$ and $\xi$ are in the interval $a\leq x\leq b$. We define

$$x_{n+1}=x_n+c_nf(x_n) \qquad (n=1, 2, \ldots).$$

Then, if $f'(x)\geq 0$ and the constants $c_n$ are negative and bounded, the sequence $x_n$ converges monotonically to the root $\xi$.

If $c_n=c=\text{constant}<0$ and $f'(x)>0$, then the process converges but not necessarily monotonically.

### Degree of Convergence of an Approximation Process

**3.9.2** Let $x_1$, $x_2$, $x_3$, . . . be an infinite sequence of approximations to a number $\xi$. Then, if

$$|x_{n+1}-\xi|<A|x_n-\xi|^k, \qquad (n=1, 2, \ldots)$$

where $A$ and $k$ are independent of $n$, the sequence is said to have convergence of at most the $k$th degree (or order or index) to $\xi$. If $k=1$ and $A<1$ the convergence is linear; if $k=2$ the convergence is quadratic.

### Regula Falsi (False Position)

**3.9.3** Given $y=f(x)$ to find $\xi$ such that $f(\xi)=0$, choose $x_0$ and $x_1$ such that $f(x_0)$ and $f(x_1)$ have opposite signs and compute

$$x_2=x_1-\frac{(x_1-x_0)}{(f_1-f_0)}\,f_1=\frac{f_1x_0-f_0x_1}{f_1-f_0}.$$

Then continue with $x_2$ and either of $x_0$ or $x_1$ for which $f(x_0)$ or $f(x_1)$ is of opposite sign to $f(x_2)$.

Regula falsi is equivalent to inverse linear interpolation.

*See page 11.

### Method of Iteration (Successive Substitution)

**3.9.4** The iteration scheme $x_{k+1}=F(x_k)$ will converge to a zero of $x=F(x)$ if

(1) $|F'(x)|\leq q<1$ for $a\leq x\leq b$,

(2) $a\leq x_0\pm\dfrac{|F(x_0)-x_0|}{1-q}\leq b.$

### Newton's Method of Successive Approximations

**3.9.5**

#### Newton's Rule

If $x=x_k$ is an approximation to the solution $x=\xi$ of $f(x)=0$ then the sequence

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

will converge quadratically to $x=\xi$: (if instead of the condition (2) above),

(1) *Monotonic convergence*, $f(x_0)f''(x_0)>0$ and $f'(x)$, $f''(x)$ do not change sign in the interval $(x_0, \xi)$, or

(2) *Oscillatory convergence*, $f(x_0)f''(x_0)<0$ and $f'(x)$, $f''(x)$ do not change sign in the interval $(x_0, x_1)$, $x_0\leq\xi\leq x_1$.

### Newton's Method Applied to Real *n*th Roots

**3.9.6** Given $x^n=N$, if $x_k$ is an approximation $x=N^{1/n}$ then the sequence

$$x_{k+1}=\frac{1}{n}\left[\frac{N}{x_k^{n-1}}+(n-1)x_k\right]$$

will converge quadratically to $x$.

If $n=2$, $x_{k+1}=\dfrac{1}{2}\left(\dfrac{N}{x_k}+x_k\right)$,

If $n=3$, $x_{k+1}=\dfrac{1}{3}\left(\dfrac{N}{x_k^2}+2x_k\right)$.

### Aitken's $\delta^2$-Process for Acceleration of Sequences

**3.9.7** If $x_k$, $x_{k+1}$, $x_{k+2}$ are three successive iterates in a sequence converging with an error which is approximately in geometric progression, then

$$\bar{x}_k=x_k-\frac{(x_k-x_{k+1})^2}{\Delta^2x_k}=\frac{x_kx_{k+2}-x_{k+1}^2}{\Delta^2x_k};$$

$$\Delta^2x_k=x_k-2x_{k+1}+x_{k+2}$$

is an improved estimate of $x$. In fact, if $x_k=x+O(\lambda^k)$ then $\bar{x}=x+O(\lambda^k)$, $|\lambda|<1$. *