Exercice 1 (Commutateur de la dérivée covariante) *Montrer que pour un tenseur arbitraire T,*

$$\begin{aligned} T^{i_1...i_p}_{j_1...j_q;kl} - T^{i_1...i_p}_{j_1...j_q;lk} &= -R^{i_1}_{mkl}T^{mi_2...i_p}_{j_1...j_q} - ... - R^{i_p}_{mkl}T^{i_1...i_{p-1}m}_{j_1...j_q} \\ &\quad +R^{m}_{j_1kl}T^{i_1...i_p}_{mj_2...j_q} + ... + R^{m}_{j_qkl}T^{i_1...i_p}_{j_1...j_{q-1}m}. \end{aligned} \quad (*)$$

Réponse à l'exercice 1

.1 Dérivé covariante d'un tenseur de type (p, q)

Premièrement, on va démontrer l'expression de la dérivée covariante pour un tenseur de type (p, q). La preuve est valide seulement pour un tenseur réductible en un produit tensoriel p de vecteurs covariants et q vecteur contravariant, ce qui constitut tout de même un début...

Soit $\mathrm{T} = T^i\partial_i$

$$\nabla_{\mathrm{X}}\mathrm{T} = T^i_{;k}X^k\partial_i,$$

où $T^i_{;k} = T^i_{,j} + T^k\Gamma^i{}_{jk}$.Soit $\mathrm{T} = \mathrm{T}^{i_1}\mathrm{T}^{i_2}...\mathrm{T}^{i_p} = T^{i_1...i_p}\partial_{i_1}...\partial_{i_p}$, avec $T^{i_1...i_p} = T^{i_1}T^{i_2}...T^{i_p}$ on a alors suivant la règle de Leibniz

$$\begin{aligned} \nabla_{\mathrm{X}}\mathrm{T} &= \left(\nabla_{\mathrm{X}}\mathrm{T}^{i_1}\right)\mathrm{T}^{i_2}...\mathrm{T}^{i_p} + \mathrm{T}^{i_1}\left(\nabla_{\mathrm{X}}\mathrm{T}^{i_2}\right)...\mathrm{T}^{i_p} \\ &\quad +... + \mathrm{T}^{i_1}\mathrm{T}^{i_2}...\left(\nabla_{\mathrm{X}}\mathrm{T}^{i_p}\right) \end{aligned}$$

Considérant un petit calcul intermédiaire pour voir où l'on va,

$$\begin{aligned} \left(\nabla_{\mathrm{X}}\mathrm{T}^{i_1}\right)\mathrm{T}^{i_2}...\mathrm{T}^{i_p} &= \left(T^{i_1}_{;k}X^k\partial_{i_1}\right)\mathrm{T}^{i_2}...\mathrm{T}^{i_p} \\ &= \left(T^{i_1}_{,j} + T^k\Gamma^{i_1}_{jk}\right)X^j\partial_{i_1}\mathrm{T}^{i_2}...\mathrm{T}^{i_p} \\ &= \left(T^{i_1}_{,j}X^j\partial_{i_1}\mathrm{T}^{i_2}...\mathrm{T}^{i_p} + \Gamma^{i_1}_{jk}T^kX^j\partial_{i_1}\mathrm{T}^{i_2}...\mathrm{T}^{i_p}\right). \end{aligned}$$

On a donc

$$\begin{aligned} \nabla_{\mathrm{X}}\mathrm{T} &= \left(\nabla_{\mathrm{X}}\mathrm{T}^{i_1}\right)\mathrm{T}^{i_2}...\mathrm{T}^{i_p} + \mathrm{T}^{i_1}\left(\nabla_{\mathrm{X}}\mathrm{T}^{i_2}\right)...\mathrm{T}^{i_p} \\ &\quad +... + \mathrm{T}^{i_1}\mathrm{T}^{i_2}...\left(\nabla_{\mathrm{X}}\mathrm{T}^{i_p}\right) \\ &= T^{i_1}_{,j}X^j\partial_{i_1}\mathrm{T}^{i_2}...\mathrm{T}^{i_p} + ... + \mathrm{T}^{i_1}\mathrm{T}^{i_2}...T^{i_p}_{,j}X^j\partial_{i_p} \\ &\quad +T^k\Gamma^{i_1}_{jk}X^j\partial_{i_1}\mathrm{T}^{i_2}...\mathrm{T}^{i_p} + ... + \mathrm{T}^{i_1}...\mathrm{T}^{i_{p-1}}T^k\Gamma^{i_p}_{jk}X^j\partial_{i_p} \\ &= \left(T^{i_1}_{,j}...T^{i_p} + T^{i_1}T^{i_2}_{,j}...T^{i_p} + ... + T^{i_1}...T^{i_p}_{,j}\right)X^j\partial_{i_1}...\partial_{i_p} \\ &\quad +\Gamma^{i_1}_{jk}T^kT^{i_2}...T^{i_p}X^j\partial_{i_1}...\partial_{i_p} + ... + \Gamma^{i_p}_{jk}T^{i_1}...T^kX^j\partial_{i_1}...\partial_{i_p} \end{aligned}$$

Et en composantes

$$T^{i_1...i_p}_{;j} = T^{i_1...i_p}_{,j} + \Gamma^{i_1}_{jk} T^{ki_2...i_p} + ... + \Gamma^{i_1}_{jk} T^{i_1...k}$$

De façon générale, puisque $T_{i;k} = T_{i,k} - \Gamma^{j}_{ik} T_j$, on a

$$\begin{aligned} T^{i_1...i_p}_{j_1...j_q;k} &= T^{i_1...i_p}_{j_1...j_q,k} + \Gamma^{i_1}_{km} T^{mi_2...i_p}_{j_1...j_q} + ... + \Gamma^{i_1}_{km} T^{i_1...m}_{j_1...j_q} \\ &\quad -\Gamma^{m}_{kj_1} T^{i_1...i_p}_{mj_2...j_q} - ... - \Gamma^{m}_{kj_q} T^{i_1...i_p}_{j_1...m}. \end{aligned} \quad (1)$$

Ce qui termine la preuve.

.2 Notation

Pour la suite, étant donné que l'écriture est très lourde, je vais introduire quelques définitions et conventions:

- Le symbole *remplacé par* "_". C'est-à-dire que a_i_m signifie "a_i *remplacé par* m".
- $\bar{a} \equiv a_1 a_2 ... a_p$. C'est-à-dire que $\bar{a}$ est une liste de caractères.
- Les crochets [] se lisent *tel que.* Exemple*:* $\bar{a}[a_2\_m]$ se lit : la liste des indices $a_1...a_p$ *tel que* a_2 *est remplacé par* m, c'est-à-dire que $\bar{a}[a_2\_m] \equiv a_1 m a_3 a_4 ... a_p$.
- Les lettres i et j sont réservées pour l'itération. Exemple :

$$\begin{aligned} \sum_{i=1}^{p} \Gamma^{a_i}_{km} T^{\bar{a}[a_i\_m]} &= \Gamma^{a_1}_{km} T^{\bar{a}[a_1\_m]} + \Gamma^{a_2}_{km} T^{\bar{a}[a_2\_m]} + ... + \Gamma^{a_p}_{km} T^{\bar{a}[a_p\_m]} \\ &= \Gamma^{a_1}_{km} T^{ma_2...a_p} + ... + \Gamma^{a_p}_{km} T^{a_1...a_{p-1}m}. \end{aligned}$$

  Si on itère de 1 à p, j'écrirai i_p (ou j_p). Puisque i_p est réservé à l'itération de 1 à p, le symbole de sommation est superflux. J'écrirai donc

$$\begin{aligned} \Gamma^{a_{i_3}}_{km} T^{\bar{a}[a_{i_3}\_m]} &= \Gamma^{a_1}_{km} T^{\bar{a}[a_1\_m]} + \Gamma^{a_2}_{km} T^{\bar{a}[a_2\_m]} + \Gamma^{a_3}_{km} T^{\bar{a}[a_3\_m]} \\ &= \Gamma^{a_1}_{km} T^{ma_2...a_p} + \Gamma^{a_2}_{km} T^{a_1 m a_3...a_p} \\ &\quad + \Gamma^{a_3}_{km} T^{a_1 a_2 m...a_p}, \end{aligned}$$

  et de façon plus générale

$$\Gamma^{a_{i_p}}_{km} T^{\bar{a}[a_{i_p}\_m]} = \Gamma^{a_1}_{km} T^{\bar{a}[a_1\_m]} + ... + \Gamma^{a_p}_{km} T^{\bar{a}[a_p\_m]}$$

Réécrite sous cette notation, l'expression (*) à démontrer s'écrit:

$$T^{\bar{a}}_{\bar{b};k\lambda} - T^{\bar{a}}_{\bar{b};\lambda k} = -R^{a_{i_p}}_{mk\lambda} T^{\bar{a}[a_{i_p}\_m]}_{\bar{b}} + R^{m}_{b_{i_q}k\lambda} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}$$

.3 Début de la preuve

Suivant cette notation, on réécrit l'équation (1) comme

$$T^{\bar{a}}_{\bar{b};k} = T^{\bar{a}}_{\bar{b},k} + \Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{i_p}\_m]}_{\bar{b}} - \Gamma^{m}_{kb_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}. \tag{2}$$

où $\bar{a} = a_1 ... a_p$ et $\bar{b} = b_1 ... b_q$. Posons maintenant

$$\begin{aligned} A^{\bar{a}}_{\bar{c}} &= A^{\bar{a}}_{k\bar{b}} = \partial_k T^{\bar{a}}_{\bar{b}} && (3)\\ B^{\bar{a}}_{\bar{c}} &= B^{\bar{a}}_{k\bar{b}} = \Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{i_p}\_m]}_{\bar{b}}, && (4)\\ C^{\bar{a}}_{\bar{c}} &= C^{\bar{a}}_{k\bar{b}} = \Gamma^{m}_{kb_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}, && (5) \end{aligned}$$

où $\bar{c} \equiv k\bar{b}$ a été introduit. L'équation (2) s'écrit maintenant comme

$$T^{\bar{a}}_{\bar{b};k} = A^{\bar{a}}_{\bar{c}} + B^{\bar{a}}_{\bar{c}} - C^{\bar{a}}_{\bar{c}}.$$

Calculons maintenant la 2e dérivée covariante :

$$\begin{aligned} \left(T^{\bar{a}}_{\bar{b};k}\right)_{;\lambda} &= \left(A^{\bar{a}}_{\bar{c}} + B^{\bar{a}}_{\bar{c}} - C^{\bar{a}}_{\bar{c}}\right)_{;\lambda} \\ &= A^{\bar{a}}_{\bar{c};\lambda} + B^{\bar{a}}_{\bar{c};\lambda} - C^{\bar{a}}_{\bar{c};\lambda} \end{aligned}$$

Utilisant l'équation (2), on a

$$\begin{aligned} T^{\bar{a}}_{\bar{b};k\lambda} = \; & A^{\bar{a}}_{\bar{c},\lambda} + \Gamma^{a_{j_p}}_{\lambda\mu} A^{\bar{a}[a_{j_p}\_\mu]}_{\bar{c}} - \Gamma^{\mu}_{\lambda c_{j_{q+1}}} A^{\bar{a}}_{\bar{c}[c_{j_{q+1}}\_\mu]} \\ & + B^{\bar{a}}_{\bar{c},\lambda} + \Gamma^{a_{j_p}}_{\lambda\mu} B^{\bar{a}[a_{j_p}\_\mu]}_{\bar{c}} - \Gamma^{\mu}_{\lambda c_{j_{q+1}}} B^{\bar{a}}_{\bar{c}[c_{j_{q+1}}\_\mu]} \\ & - \left(C^{\bar{a}}_{\bar{c},\lambda} + \Gamma^{a_{j_p}}_{\lambda\mu} C^{\bar{a}[a_{j_p}\_\mu]}_{\bar{c}} - \Gamma^{\mu}_{\lambda c_{j_{q+1}}} C^{\bar{a}}_{\bar{c}[c_{j_{q+1}}\_\mu]} \right), \end{aligned}$$

On a donc, de la même façon,

$$T^{\bar{a}}_{\bar{b};\lambda k} = D^{\bar{a}}_{\bar{d};k} + E^{\bar{a}}_{\bar{d};k} - F^{\bar{a}}_{\bar{d};k},$$

c'est-à-dire,

$$\begin{aligned} T^{\bar{a}}_{\bar{b};\lambda k} = \; & D^{\bar{a}}_{\bar{d},k} + \Gamma^{a_{j_p}}_{km} D^{\bar{a}[a_{j_p}\_m]}_{\bar{d}} - \Gamma^{m}_{kd_{j_{q+1}}} D^{\bar{a}}_{\bar{d}[d_{j_{q+1}}\_m]} \\ & + E^{\bar{a}}_{\bar{d},k} + \Gamma^{a_{j_p}}_{km} E^{\bar{a}[a_{j_p}\_m]}_{\bar{d}} - \Gamma^{m}_{kd_{j_{q+1}}} E^{\bar{a}}_{\bar{d}[d_{j_{q+1}}\_m]} \\ & - \left(F^{\bar{a}}_{\bar{d},k} + \Gamma^{a_{j_p}}_{km} F^{\bar{a}[a_{j_p}\_m]}_{\bar{d}} - \Gamma^{m}_{kd_{j_{q+1}}} F^{\bar{a}}_{\bar{d}[d_{j_{q+1}}\_m]} \right) \end{aligned}$$

où $\bar{d} = \lambda\bar{b}$ a été défini, et où

$$
\begin{aligned}
D^{\bar{a}}_{\bar{d}} &= D^{\bar{a}}_{\lambda\bar{b}} = \partial_\lambda T^{\bar{a}}_{\bar{b}}, && (6)\\
E^{\bar{a}}_{\bar{d}} &= E^{\bar{a}}_{\lambda\bar{b}} = \Gamma^{a_{ip}}_{\lambda\mu} T^{\bar{a}[a_{i_p}-\mu]}_{\bar{b}}, && (7)\\
F^{\bar{a}}_{\bar{d}} &= F^{\bar{a}}_{\lambda\bar{b}} = \Gamma^{\mu}_{\lambda b_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}-\mu]}. && (8)
\end{aligned}
$$

Ainsi,

$$
\begin{aligned}
T^{\bar{a}}_{\bar{b};k\lambda} - T^{\bar{a}}_{\bar{b};\lambda k} &= A^{\bar{a}}_{\bar{c};\lambda} + B^{\bar{a}}_{\bar{c};\lambda} - C^{\bar{a}}_{\bar{c};\lambda}\\
&\quad - D^{\bar{a}}_{\bar{d};k} - E^{\bar{a}}_{\bar{d};k} + F^{\bar{a}}_{\bar{d};k}\\
&= A^{\bar{a}}_{\bar{c},\lambda} + \Gamma^{a_{j_p}}_{\lambda\mu} A^{\bar{a}[a_{j_p}-\mu]}_{\bar{c}} - \Gamma^{\mu}_{\lambda c_{j_{q+1}}} A^{\bar{a}}_{\bar{c}[c_{j_{q+1}}-\mu]}\\
&\quad + B^{\bar{a}}_{\bar{c},\lambda} + \Gamma^{a_{j_p}}_{\lambda\mu} B^{\bar{a}[a_{j_p}-\mu]}_{\bar{c}} - \Gamma^{\mu}_{\lambda c_{j_{q+1}}} B^{\bar{a}}_{\bar{c}[c_{j_{q+1}}-\mu]}\\
&\quad - C^{\bar{a}}_{\bar{c},\lambda} - \Gamma^{a_{j_p}}_{\lambda\mu} C^{\bar{a}[a_{j_p}-\mu]}_{\bar{c}} + \Gamma^{\mu}_{\lambda c_{j_{q+1}}} C^{\bar{a}}_{\bar{c}[c_{j_{q+1}}-\mu]}\\
&\quad - D^{\bar{a}}_{\bar{d},k} - \Gamma^{a_{j_p}}_{km} D^{\bar{a}[a_{j_p}-m]}_{\bar{d}} + \Gamma^{m}_{kd_{j_{q+1}}} D^{\bar{a}}_{\bar{d}[d_{j_{q+1}}-m]}\\
&\quad - E^{\bar{a}}_{\bar{d},k} - \Gamma^{a_{j_p}}_{km} E^{\bar{a}[a_{j_p}-m]}_{\bar{d}} + \Gamma^{m}_{kd_{j_{q+1}}} E^{\bar{a}}_{\bar{d}[d_{j_{q+1}}-m]}\\
&\quad + F^{\bar{a}}_{\bar{d},k} + \Gamma^{a_{j_p}}_{km} F^{\bar{a}[a_{j_p}-m]}_{\bar{d}} - \Gamma^{m}_{kd_{j_{q+1}}} F^{\bar{a}}_{\bar{d}[d_{j_{q+1}}-m]}
\end{aligned}
$$

Regroupant les termes, sans aucune simplification, on obtient

$$
\begin{aligned}
T^{\bar{a}}_{\bar{b};k\lambda} - T^{\bar{a}}_{\bar{b};\lambda k} &= A^{\bar{a}}_{\bar{c},\lambda} + B^{\bar{a}}_{\bar{c},\lambda} - C^{\bar{a}}_{\bar{c},\lambda} - D^{\bar{a}}_{\bar{d},k} - E^{\bar{a}}_{\bar{d},k} + F^{\bar{a}}_{\bar{d},k} && \text{(i)}\\
&\quad + \Gamma^{a_{j_p}}_{\lambda\mu}\left(A^{\bar{a}[a_{j_p}-\mu]}_{\bar{c}} + B^{\bar{a}[a_{j_p}-\mu]}_{\bar{c}} - C^{\bar{a}[a_{j_p}-\mu]}_{\bar{c}}\right) && \text{(ii)}\\
&\quad - \Gamma^{\mu}_{\lambda c_{j_{q+1}}}\left(A^{\bar{a}}_{\bar{c}[c_{j_{q+1}}-\mu]} + B^{\bar{a}}_{\bar{c}[c_{j_{q+1}}-\mu]} - C^{\bar{a}}_{\bar{c}[c_{j_{q+1}}-\mu]}\right) && \text{(iii)}\\
&\quad - \Gamma^{a_{j_p}}_{km}\left(D^{\bar{a}[a_{j_p}-m]}_{\bar{d}} + E^{\bar{a}[a_{j_p}-m]}_{\bar{d}} - F^{\bar{a}[a_{j_p}-m]}_{\bar{d}}\right) && \text{(iv)}\\
&\quad + \Gamma^{m}_{kd_{j_{q+1}}}\left(D^{\bar{a}}_{\bar{d}[d_{j_{q+1}}-m]} + E^{\bar{a}}_{\bar{d}[d_{j_{q+1}}-m]} - F^{\bar{a}}_{\bar{d}[d_{j_{q+1}}-m]}\right) && \text{(v)}
\end{aligned}
$$

On va maintenant travailler ligne par ligne. Pour la première

$$
\begin{aligned}
\text{(i)} &= A^{\bar{a}}_{\bar{c},\lambda} + B^{\bar{a}}_{\bar{c},\lambda} - C^{\bar{a}}_{\bar{c},\lambda} - D^{\bar{a}}_{\bar{d},k} - E^{\bar{a}}_{\bar{d},k} + F^{\bar{a}}_{\bar{d},k} \\
&= \left(\partial_k T^{\bar{a}}_{\bar{b}}\right)_{,\lambda} + \left(\Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}}\right)_{,\lambda} - \left(\Gamma^{m}_{kb_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_m]}\right)_{,\lambda} \\
&\quad - \left(\partial_\lambda T^{\bar{a}}_{\bar{b}}\right)_{,k} - \left(\Gamma^{a_{ip}}_{\lambda\mu} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b}}\right)_{,k} + \left(\Gamma^{\mu}_{\lambda b_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_\mu]}\right)_{,k} \\
&= \left(\Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}}\right)_{,\lambda} - \left(\Gamma^{m}_{kb_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_m]}\right)_{,\lambda} \\
&\quad - \left(\Gamma^{a_{ip}}_{\lambda\mu} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b}}\right)_{,k} + \left(\Gamma^{\mu}_{\lambda b_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_\mu]}\right)_{,k} \\
&= \Gamma^{a_{ip}}_{km,\lambda} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}} + \Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b},\lambda} - \Gamma^{m}_{kb_{iq},\lambda} T^{\bar{a}}_{\bar{b}[b_{iq}\_m]} - \Gamma^{m}_{kb_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_m],\lambda} \\
&\quad -\Gamma^{a_{ip}}_{\lambda\mu,k} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b}} - \Gamma^{a_{ip}}_{\lambda\mu} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b},k} + \Gamma^{\mu}_{\lambda b_{iq},k} T^{\bar{a}}_{\bar{b}[b_{iq}\_\mu]} + \Gamma^{\mu}_{\lambda b_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_\mu],k} \\
&= \Gamma^{a_{ip}}_{km,\lambda} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}} - \Gamma^{a_{ip}}_{\lambda\mu,k} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b}} + \Gamma^{\mu}_{\lambda b_{iq},k} T^{\bar{a}}_{\bar{b}[b_{iq}\_\mu]} - \Gamma^{m}_{kb_{iq},\lambda} T^{\bar{a}}_{\bar{b}[b_{iq}\_m]} \\
&\quad +\Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b},\lambda} - \Gamma^{a_{ip}}_{\lambda\mu} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b},k} + \Gamma^{\mu}_{\lambda b_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_\mu],k} - \Gamma^{m}_{kb_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_m],\lambda}
\end{aligned}
$$

Renommant l'indice muet μ par m, on obtient

$$
\begin{aligned}
\text{(i)} &= \left(\Gamma^{a_{ip}}_{km,\lambda} - \Gamma^{a_{ip}}_{\lambda m,k}\right) T^{\bar{a}[a_{ip}\_m]}_{\bar{b}} + \left(\Gamma^{m}_{\lambda b_{iq},k} - \Gamma^{m}_{kb_{iq},\lambda}\right) T^{\bar{a}}_{\bar{b}[b_{iq}\_m]} \\
&\quad +\Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b},\lambda} - \Gamma^{a_{ip}}_{\lambda m} T^{\bar{a}[a_{ip}\_m]}_{\bar{b},k} + \Gamma^{m}_{\lambda b_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_m],k} - \Gamma^{m}_{kb_{iq}} T^{\bar{a}}_{\bar{b}[b_{iq}\_m],\lambda}
\end{aligned}
$$

Maintenant la ligne (ii), substituant A, B et C des équations (3), (4) et (5), on obtient

$$
\begin{aligned}
\text{(ii)} &= \Gamma^{a_{jp}}_{\lambda\mu} \left(A^{\bar{a}[a_{jp}\_\mu]}_{\bar{c}} + B^{\bar{a}[a_{jp}\_\mu]}_{\bar{c}} - C^{\bar{a}[a_{jp}\_\mu]}_{\bar{c}}\right) \\
&= \Gamma^{a_{jp}}_{\lambda\mu} \partial_k T^{\bar{a}[a_{jp}\_\mu]}_{\bar{b}} - \Gamma^{a_{jp}}_{\lambda\mu} \Gamma^{m}_{kb_{iq}} T^{\bar{a}[a_{jp}\_\mu]}_{\bar{b}[b_{iq}\_m]} \\
&\quad +\Gamma^{a_{jp}}_{\lambda\mu} \left(\Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}}\right)^{[a_{jp}\_\mu]}
\end{aligned}
$$

Maintenant la ligne (iii),

$$
\begin{aligned}
\text{(iii)} &= -\Gamma^{\mu}_{\lambda c_{j_{q+1}}}\left(A^{\bar{a}}_{\bar{c}[c_{j_{q+1}}\_\mu]} + B^{\bar{a}}_{\bar{c}[c_{j_{q+1}}\_\mu]} - C^{\bar{a}}_{\bar{c}[c_{j_{q+1}}\_\mu]}\right) \\
&= -\Gamma^{\mu}_{\lambda c_1}\left(A^{\bar{a}}_{\bar{c}[c_1\_\mu]} + B^{\bar{a}}_{\bar{c}[c_1\_\mu]} - C^{\bar{a}}_{\bar{c}[c_1\_\mu]}\right) \\
&\quad -\Gamma^{\mu}_{\lambda c_{j_q}}\left(A^{\bar{a}}_{\bar{c}[c_{j_q}\_\mu]} + B^{\bar{a}}_{\bar{c}[c_{j_q}\_\mu]} - C^{\bar{a}}_{\bar{c}[c_{j_q}\_\mu]}\right) \qquad (j_q = 2...q+1) \\
&= -\Gamma^{\mu}_{\lambda k}\left(A^{\bar{a}}_{\bar{c}[k\_\mu]} + B^{\bar{a}}_{\bar{c}[k\_\mu]} - C^{\bar{a}}_{\bar{c}[k\_\mu]}\right) \qquad (c_1 = k) \\
&\quad -\Gamma^{\mu}_{\lambda b_{j_q}}\left(A^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]} + B^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]} - C^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]}\right) \qquad \left(j_q = 1...q;\ \bar{c} = k\bar{b}\right) \\
&= -\Gamma^{\mu}_{\lambda k}\left(A^{\bar{a}}_{\{k\bar{b}\}[k\_\mu]} + B^{\bar{a}}_{\{k\bar{b}\}[k\_\mu]} - C^{\bar{a}}_{\{k\bar{b}\}[k\_\mu]}\right) \\
&\quad -\Gamma^{\mu}_{\lambda b_{j_q}}\left(A^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]} + B^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]} - C^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]}\right) \\
&= -\Gamma^{\mu}_{\lambda k}\left(A^{\bar{a}}_{\mu\bar{b}} + B^{\bar{a}}_{\mu\bar{b}} - C^{\bar{a}}_{\mu\bar{b}}\right) \\
&\quad -\Gamma^{\mu}_{\lambda b_{j_q}}\left(A^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]} + B^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]} - C^{\bar{a}}_{k\bar{b}[b_{j_q}\_\mu]}\right)
\end{aligned}
$$

Substituant A, B et C des équations (3), (4) et (5), on obtient

$$
\begin{aligned}
\text{(iii)} &= -\Gamma^{\mu}_{\lambda k}\left(\partial_\mu T^{\bar{a}}_{\bar{b}} + \Gamma^{a_{ip}}_{\mu m} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}} - \Gamma^{m}_{\mu b_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right) \\
&\quad -\Gamma^{\mu}_{\lambda b_{j_q}}\left(\partial_k T^{\bar{a}}_{\bar{b}[b_{j_q}\_\mu]} + \Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}[b_{j_q}\_\mu]} - \left(\Gamma^{m}_{kb_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]}\right) \\
&= -\Gamma^{\mu}_{\lambda k}\partial_\mu T^{\bar{a}}_{\bar{b}} - \Gamma^{\mu}_{\lambda k}\Gamma^{a_{ip}}_{\mu m} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}} + \Gamma^{\mu}_{\lambda k}\Gamma^{m}_{\mu b_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]} \\
&\quad -\Gamma^{\mu}_{\lambda b_{j_q}}\partial_k T^{\bar{a}}_{\bar{b}[b_{j_q}\_\mu]} - \Gamma^{\mu}_{\lambda b_{j_q}}\Gamma^{a_{ip}}_{km} T^{\bar{a}[a_{ip}\_m]}_{\bar{b}[b_{j_q}\_\mu]} + \Gamma^{\mu}_{\lambda b_{j_q}}\left(\Gamma^{m}_{kb_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]}
\end{aligned}
$$

Maintenant la ligne (iv), substituant simplement D, E et F des équations (6), (7) et (8)

$$
\begin{aligned}
\text{(iv)} &= -\Gamma^{a_{j_p}}_{km}\left(D^{\bar{a}[a_{j_p}\_m]}_{\bar{d}} + E^{\bar{a}[a_{j_p}\_m]}_{\bar{d}} - F^{\bar{a}[a_{j_p}\_m]}_{\bar{d}}\right) \\
&= -\Gamma^{a_{j_p}}_{km}\partial_\lambda T^{\bar{a}[a_{j_p}\_m]}_{\bar{b}} + \Gamma^{a_{j_p}}_{km}\Gamma^{\mu}_{\lambda b_{i_q}} T^{\bar{a}[a_{j_p}\_m]}_{\bar{b}[b_{i_q}\_\mu]} \\
&\quad -\Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{ip}}_{\lambda\mu} T^{\bar{a}[a_{ip}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]}
\end{aligned}
$$

Maintenant la ligne (v) :

$$
\begin{aligned}
\text{(v)} &= \Gamma^m_{kd_{j_{q+1}}}\left(D^{\bar a}_{\bar d[d_{j_{q+1}}\_m]} + E^{\bar a}_{\bar d[d_{j_{q+1}}\_m]} - F^{\bar a}_{\bar d[d_{j_{q+1}}\_m]}\right)\\
&= \Gamma^m_{kd_1}\left(D^{\bar a}_{\bar d[d_1\_m]} + E^{\bar a}_{\bar d[d_1\_m]} - F^{\bar a}_{\bar d[d_1\_m]}\right)\\
&\quad +\Gamma^m_{kd_{j_q}}\left(D^{\bar a}_{\bar d[d_{j_q}\_m]} + E^{\bar a}_{\bar d[d_{j_q}\_m]} - F^{\bar a}_{\bar d[d_{j_q}\_m]}\right) \qquad (j_q = 2...q+1)\\
&= \Gamma^m_{k\lambda}\left(D^{\bar a}_{\{\lambda\bar b\}[\lambda\_m]} + E^{\bar a}_{\{\lambda\bar b\}[\lambda\_m]} - F^{\bar a}_{\{\lambda\bar b\}[\lambda\_m]}\right) \qquad \left(d_1 = \lambda\,;\bar d = \lambda\bar b\right)\\
&\quad +\Gamma^m_{kb_{j_q}}\left(D^{\bar a}_{\lambda\bar b[b_{j_q}\_m]} + E^{\bar a}_{\lambda\bar b[b_{j_q}\_m]} - F^{\bar a}_{\lambda\bar b[b_{j_q}\_m]}\right)\\
&= \Gamma^m_{k\lambda}\left(D^{\bar a}_{m\bar b} + E^{\bar a}_{m\bar b} - F^{\bar a}_{m\bar b}\right)\\
&\quad +\Gamma^m_{kb_{j_q}}\left(D^{\bar a}_{\lambda\bar b[b_{j_q}\_m]} + E^{\bar a}_{\lambda\bar b[b_{j_q}\_m]} - F^{\bar a}_{\lambda\bar b[b_{j_q}\_m]}\right)
\end{aligned}
$$

Substituant simplement D, E et F des équations (6), (7) et (8), on obtient

$$
\begin{aligned}
\text{(v)} &= \Gamma^m_{k\lambda}\left(\partial_m T^{\bar a}_{\bar b} + \Gamma^{a_{ip}}_{m\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b} - \Gamma^{\mu}_{mb_{iq}} T^{\bar a}_{\bar b[b_{iq}\_\mu]}\right)\\
&\quad +\Gamma^m_{kb_{j_q}}\left(\partial_\lambda T^{\bar a}_{\bar b[b_{j_q}\_m]} + \Gamma^{a_{ip}}_{\lambda\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b[b_{j_q}\_m]} - \left(\Gamma^{\mu}_{\lambda b_{iq}} T^{\bar a}_{\bar b[b_{iq}\_\mu]}\right)_{[b_{j_q}\_m]}\right)\\
&= \Gamma^m_{k\lambda}\partial_m T^{\bar a}_{\bar b} + \Gamma^m_{k\lambda}\Gamma^{a_{ip}}_{m\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b} - \Gamma^m_{k\lambda}\Gamma^{\mu}_{mb_{iq}} T^{\bar a}_{\bar b[b_{iq}\_\mu]}\\
&\quad +\Gamma^m_{kb_{j_q}}\partial_\lambda T^{\bar a}_{\bar b[b_{j_q}\_m]} + \Gamma^m_{kb_{j_q}}\Gamma^{a_{ip}}_{\lambda\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b[b_{j_q}\_m]} - \Gamma^m_{kb_{j_q}}\left(\Gamma^{\mu}_{\lambda b_{iq}} T^{\bar a}_{\bar b[b_{iq}\_\mu]}\right)_{[b_{j_q}\_m]}
\end{aligned}
$$

Substituant les "simplifications" de (i) à (v) dans $T^{\bar a}_{\bar b;k\lambda} - T^{\bar a}_{\bar b;\lambda k}$, on trouve

$$
\begin{aligned}
T^{\bar a}_{\bar b;k\lambda} - T^{\bar a}_{\bar b;\lambda k} &= \left(\Gamma^{a_{ip}}_{km,\lambda} - \Gamma^{a_{ip}}_{\lambda m,k}\right) T^{\bar a[a_{ip}\_m]}_{\bar b} + \left(\Gamma^m_{\lambda b_{iq},k} - \Gamma^m_{kb_{iq},\lambda}\right) T^{\bar a}_{\bar b[b_{iq}\_m]} && \text{(i)}\\
&\quad +\Gamma^{a_{ip}}_{km} T^{\bar a[a_{ip}\_m]}_{\bar b,\lambda} - \Gamma^{a_{ip}}_{\lambda m} T^{\bar a[a_{ip}\_m]}_{\bar b,k} + \Gamma^m_{\lambda b_{iq}} T^{\bar a}_{\bar b[b_{iq}\_m],k} - \Gamma^m_{kb_{iq}} T^{\bar a}_{\bar b[b_{iq}\_m],\lambda} && \text{(i)}\\
&\quad +\Gamma^{a_{jp}}_{\lambda\mu} T^{\bar a[a_{jp}\_\mu]}_{\bar b,k} - \Gamma^{a_{jp}}_{\lambda\mu}\Gamma^m_{kb_{iq}} T^{\bar a[a_{jp}\_\mu]}_{\bar b[b_{iq}\_m]} + \Gamma^{a_{jp}}_{\lambda\mu}\left(\Gamma^{a_{ip}}_{km} T^{\bar a[a_{ip}\_m]}_{\bar b}\right)^{[a_{jp}\_\mu]} && \text{(ii)}\\
&\quad -\Gamma^{\mu}_{\lambda k} T^{\bar a}_{\bar b,\mu} - \Gamma^{\mu}_{\lambda k}\Gamma^{a_{ip}}_{\mu m} T^{\bar a[a_{ip}\_m]}_{\bar b} + \Gamma^{\mu}_{\lambda k}\Gamma^m_{\mu b_{iq}} T^{\bar a}_{\bar b[b_{iq}\_m]} && \text{(iii)}\\
&\quad -\Gamma^{\mu}_{\lambda b_{jq}} T^{\bar a}_{\bar b[b_{jq}\_\mu],k} - \Gamma^{\mu}_{\lambda b_{jq}}\Gamma^{a_{ip}}_{km} T^{\bar a[a_{ip}\_m]}_{\bar b[b_{jq}\_\mu]} + \Gamma^{\mu}_{\lambda b_{jq}}\left(\Gamma^m_{kb_{iq}} T^{\bar a}_{\bar b[b_{iq}\_m]}\right)_{[b_{jq}\_\mu]} && \text{(iii)}\\
&\quad -\Gamma^{a_{jp}}_{km} T^{\bar a[a_{jp}\_m]}_{\bar b,\lambda} + \Gamma^{a_{jp}}_{km}\Gamma^{\mu}_{\lambda b_{iq}} T^{\bar a[a_{jp}\_m]}_{\bar b[b_{iq}\_\mu]} - \Gamma^{a_{jp}}_{km}\left(\Gamma^{a_{ip}}_{\lambda\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b}\right)^{[a_{jp}\_m]} && \text{(iv)}\\
&\quad +\Gamma^m_{k\lambda} T^{\bar a}_{\bar b,m} + \Gamma^m_{k\lambda}\Gamma^{a_{ip}}_{m\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b} - \Gamma^m_{k\lambda}\Gamma^{\mu}_{mb_{iq}} T^{\bar a}_{\bar b[b_{iq}\_\mu]} && \text{(v)}\\
&\quad +\Gamma^m_{kb_{jq}} T^{\bar a}_{\bar b[b_{jq}\_m],\lambda} + \Gamma^m_{kb_{jq}}\Gamma^{a_{ip}}_{\lambda\mu} T^{\bar a[a_{ip}\_\mu]}_{\bar b[b_{jq}\_m]} - \Gamma^m_{kb_{jq}}\left(\Gamma^{\mu}_{\lambda b_{iq}} T^{\bar a}_{\bar b[b_{iq}\_\mu]}\right)_{[b_{jq}\_m]} && \text{(v)}
\end{aligned}
$$

où le numéro de la ligne de provenance est clairement indiqué. En réarrangeant les termes sans en annuler aucun, de façon à ce que ceux qui s'annulent soient mis l'un à côté de l'autre et en renommant quelques indices muets, on trouve

$$
\begin{aligned}
T^{\bar a}_{\bar b;k\lambda} - T^{\bar a}_{\bar b;\lambda k} = & \left(\Gamma^{a_{i_p}}_{km,\lambda} - \Gamma^{a_{i_p}}_{\lambda m,k} - \Gamma^{\mu}_{\lambda k}\Gamma^{a_{i_p}}_{\mu m} + \Gamma^{\mu}_{k\lambda}\Gamma^{a_{i_p}}_{\mu m}\right) T^{\bar a[a_{i_p}\_m]}_{\bar b} \\
& + \left(\Gamma^{m}_{\lambda b_{i_q},k} - \Gamma^{m}_{k b_{i_q},\lambda} + \Gamma^{\mu}_{\lambda k}\Gamma^{m}_{\mu b_{i_q}} - \Gamma^{\mu}_{k\lambda}\Gamma^{m}_{\mu b_{i_q}}\right) T^{\bar a}_{\bar b[b_{i_q}\_m]} \\
& + \Gamma^{a_{i_p}}_{km} T^{\bar a[a_{i_p}\_m]}_{\bar b,\lambda} - \Gamma^{a_{j_p}}_{km} T^{\bar a[a_{j_p}\_m]}_{\bar b,\lambda} \\
& + \Gamma^{a_{j_p}}_{\lambda\mu} T^{\bar a[a_{j_p}\_\mu]}_{\bar b,k} - \Gamma^{a_{i_p}}_{\lambda m} T^{\bar a[a_{i_p}\_m]}_{\bar b,k} \\
& + \Gamma^{m}_{k\lambda} T^{\bar a}_{\bar b,m} - \Gamma^{\mu}_{\lambda k} T^{\bar a}_{\bar b,\mu} + \Gamma^{m}_{k b_{j_q}} T^{\bar a}_{\bar b[b_{j_q}\_m],\lambda} - \Gamma^{m}_{k b_{i_q}} T^{\bar a}_{\bar b[b_{i_q}\_m],\lambda} \\
& + \Gamma^{m}_{\lambda b_{i_q}} T^{\bar a}_{\bar b[b_{i_q}\_m],k} - \Gamma^{\mu}_{\lambda b_{j_q}} T^{\bar a}_{\bar b[b_{j_q}\_\mu],k} \\
& + \left(\Gamma^{a_{j_p}}_{km}\Gamma^{\mu}_{\lambda b_{i_q}} - \Gamma^{a_{i_p}}_{km}\Gamma^{\mu}_{\lambda b_{j_q}} - \Gamma^{a_{j_p}}_{\lambda m}\Gamma^{\mu}_{k b_{i_q}} + \Gamma^{a_{i_p}}_{\lambda m}\Gamma^{\mu}_{k b_{j_q}}\right) T^{\bar a[a_{i_p}\_m]}_{\bar b[b_{j_q}\_\mu]} \\
& + \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_{i_p}}_{k\mu} T^{\bar a[a_{i_p}\_\mu]}_{\bar b}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{i_p}}_{\lambda\mu} T^{\bar a[a_{i_p}\_\mu]}_{\bar b}\right)^{[a_{j_p}\_m]} \\
& + \Gamma^{\mu}_{\lambda b_{j_q}}\left(\Gamma^{m}_{k b_{i_q}} T^{\bar a}_{\bar b[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} - \Gamma^{\mu}_{k b_{j_q}}\left(\Gamma^{m}_{\lambda b_{i_q}} T^{\bar a}_{\bar b[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]}
\end{aligned}
$$

En supprimant les termes qui s'annulent (certains itérateurs j_p sont renommés i_p), on trouve

$$
\begin{aligned}
T^{\bar a}_{\bar b;k\lambda} - T^{\bar a}_{\bar b;\lambda k} = & \left(\Gamma^{a_{i_p}}_{km,\lambda} - \Gamma^{a_{i_p}}_{\lambda m,k} - \Gamma^{\mu}_{\lambda k}\Gamma^{a_{i_p}}_{\mu m} + \Gamma^{\mu}_{k\lambda}\Gamma^{a_{i_p}}_{\mu m}\right) T^{\bar a[a_{i_p}\_m]}_{\bar b} \\
& + \left(\Gamma^{m}_{\lambda b_{i_q},k} - \Gamma^{m}_{k b_{i_q},\lambda} + \Gamma^{\mu}_{\lambda k}\Gamma^{m}_{\mu b_{i_q}} - \Gamma^{\mu}_{k\lambda}\Gamma^{m}_{\mu b_{i_q}}\right) T^{\bar a}_{\bar b[b_{i_q}\_m]} \\
& + \Gamma^{m}_{k\lambda} T^{\bar a}_{\bar b,m} - \Gamma^{\mu}_{\lambda k} T^{\bar a}_{\bar b,\mu} + \mathrm{G}
\end{aligned}
$$

où

$$
\begin{aligned}
\mathrm{G} = & \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_{i_p}}_{k\mu} T^{\bar a[a_{i_p}\_\mu]}_{\bar b}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{i_p}}_{\lambda\mu} T^{\bar a[a_{i_p}\_\mu]}_{\bar b}\right)^{[a_{j_p}\_m]} \\
& + \Gamma^{\mu}_{\lambda b_{j_q}}\left(\Gamma^{m}_{k b_{i_q}} T^{\bar a}_{\bar b[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} - \Gamma^{\mu}_{k b_{j_q}}\left(\Gamma^{m}_{\lambda b_{i_q}} T^{\bar a}_{\bar b[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]}
\end{aligned}
$$

Posant $C^{l}_{k\lambda} = \Gamma^{l}_{k\lambda} - \Gamma^{l}_{\lambda k}$, on trouve

$$
\begin{aligned}
T^{\bar a}_{\bar b;k\lambda} - T^{\bar a}_{\bar b;\lambda k} = & \left(\Gamma^{a_{i_p}}_{km,\lambda} - \Gamma^{a_{i_p}}_{\lambda m,k} - C^{\mu}_{\lambda k}\Gamma^{a_{i_p}}_{\mu m}\right) T^{\bar a[a_{i_p}\_m]}_{\bar b} \\
& + \left(\Gamma^{m}_{\lambda b_{i_q},k} - \Gamma^{m}_{k b_{i_q},\lambda} - C^{\mu}_{k\lambda}\Gamma^{m}_{\mu b_{i_q}}\right) T^{\bar a}_{\bar b[b_{i_q}\_m]} \\
& + C^{m}_{k\lambda} T^{\bar a}_{\bar b,m} + \mathrm{G1} + \mathrm{G2}
\end{aligned}
$$

Où G1 et G2 sont données par

$$\begin{aligned} \mathrm{G1} &= \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_{i_p}}_{k\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{i_p}}_{\lambda\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} \\ \mathrm{G2} &= \Gamma^{\mu}_{\lambda b_{j_q}}\left(\Gamma^{m}_{k b_{i_q}}T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} - \Gamma^{\mu}_{k b_{j_q}}\left(\Gamma^{m}_{\lambda b_{i_q}}T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} \end{aligned}$$

On va maintenant montrer par induction que l'expression suivante, qui correspond à G1, est vraie :

$$\mathrm{P}(p) : \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_{i_p}}_{k\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{i_p}}_{\lambda\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} = \left(\Gamma^{a_{j_p}}_{\lambda m}\Gamma^{m}_{k\mu} - \Gamma^{a_{j_p}}_{km}\Gamma^{m}_{\lambda\mu}\right)T^{\bar{a}[a_{j_p}\_\mu]}_{\bar{b}}.$$

Pour $p = 1$:

$$\begin{aligned} &\Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_{i_p}}_{k\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{i_p}}_{\lambda\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} \\ &= \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_1}_{k\mu}T^{\mu}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_1}_{\lambda\mu}T^{\mu}_{\bar{b}}\right)^{[a_{j_p}\_m]} \\ &= \left(\Gamma^{a_1}_{\lambda m}\Gamma^{m}_{k\mu} - \Gamma^{a_1}_{\lambda m}\Gamma^{m}_{\lambda\mu}\right)T^{\mu}_{\bar{b}} \end{aligned}$$

On a montré que $\mathrm{P}(1)$ était vrai ($\mathrm{P}(2)$ l'est aussi[1]). Supposons que $\mathrm{P}(p)$ l'est et vérifions $\mathrm{P}(p+1)$:

$$\begin{aligned} &\mathrm{P}(p+1) = \Gamma^{a_{j_{p+1}}}_{\lambda m}\left(\Gamma^{a_{i_{p+1}}}_{k\mu}T^{\bar{a}[a_{i_{p+1}}\_\mu]}_{\bar{b}}\right)^{[a_{j_{p+1}}\_m]} - \Gamma^{a_{j_{p+1}}}_{km}\left(\Gamma^{a_{i_{p+1}}}_{\lambda\mu}T^{\bar{a}[a_{i_{p+1}}\_\mu]}_{\bar{b}}\right)^{[a_{j_{p+1}}\_m]} \\ &= \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_{i_p}}_{k\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} + \Gamma^{a_{p+1}}_{\lambda m}\left(\Gamma^{a_{p+1}}_{k\mu}T^{\bar{a}[a_{p+1}\_\mu]}_{\bar{b}}\right)^{[a_{p+1}\_m]} \\ &-\Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_{i_p}}_{\lambda\mu}T^{\bar{a}[a_{i_p}\_\mu]}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{p+1}}_{km}\left(\Gamma^{a_{p+1}}_{\lambda\mu}T^{\bar{a}[a_{p+1}\_\mu]}_{\bar{b}}\right)^{[a_{p+1}\_m]} \\ &= \left(\Gamma^{a_{j_p}}_{\lambda m}\Gamma^{m}_{k\mu} - \Gamma^{a_{j_p}}_{km}\Gamma^{m}_{\lambda\mu}\right)T^{\bar{a}[a_{j_p}\_\mu]}_{\bar{b}} + \Gamma^{a_{p+1}}_{\lambda m}\left(\Gamma^{m}_{k\mu}T^{\bar{a}[a_{p+1}\_\mu]}_{\bar{b}}\right) - \Gamma^{a_{p+1}}_{km}\left(\Gamma^{m}_{\lambda\mu}T^{\bar{a}[a_{p+1}\_\mu]}_{\bar{b}}\right) \\ &= \left(\Gamma^{a_{j_p}}_{\lambda m}\Gamma^{m}_{k\mu} - \Gamma^{a_{j_p}}_{km}\Gamma^{m}_{\lambda\mu}\right)T^{\bar{a}[a_{j_p}\_\mu]}_{\bar{b}} + \left(\Gamma^{a_{p+1}}_{\lambda m}\Gamma^{m}_{k\mu} - \Gamma^{a_{p+1}}_{km}\Gamma^{m}_{\lambda\mu}\right)T^{\bar{a}[a_{p+1}\_\mu]}_{\bar{b}} \end{aligned}$$

[1] Pour $p = 2$:

$$\begin{aligned} &\Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_1}_{k\mu}T^{\mu a_2}_{\bar{b}} + \Gamma^{a_2}_{k\mu}T^{a_1\mu}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_1}_{\lambda\mu}T^{\mu a_2}_{\bar{b}} + \Gamma^{a_2}_{\lambda\mu}T^{a_1\mu}_{\bar{b}}\right)^{[a_{j_p}\_m]} \\ &= \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_1}_{k\mu}T^{\mu a_2}_{\bar{b}}\right)^{[a_{j_p}\_m]} + \Gamma^{a_{j_p}}_{\lambda m}\left(\Gamma^{a_2}_{k\mu}T^{a_1\mu}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_1}_{\lambda\mu}T^{\mu a_2}_{\bar{b}}\right)^{[a_{j_p}\_m]} - \Gamma^{a_{j_p}}_{km}\left(\Gamma^{a_2}_{\lambda\mu}T^{a_1\mu}_{\bar{b}}\right)^{[a_{j_p}\_m]} \\ &= \Gamma^{a1}_{\lambda m}\Gamma^{m}_{k\mu}T^{\mu a_2}_{\bar{b}} - \Gamma^{a_1}_{km}\Gamma^{m}_{\lambda\mu}T^{\mu a_2}_{\bar{b}} + \Gamma^{a_2}_{\lambda m}\Gamma^{m}_{k\mu}T^{a_1\mu}_{\bar{b}} - \Gamma^{a_2}_{km}\Gamma^{m}_{\lambda\mu}T^{a_1\mu}_{\bar{b}} \\ &+\underbrace{\Gamma^{a1}_{\lambda m}\Gamma^{a_2}_{k\mu}T^{m\mu}_{\bar{b}} - \Gamma^{a_2}_{km}\Gamma^{a_1}_{\lambda\mu}T^{\mu m}_{\bar{b}}}_{=0} + \underbrace{\Gamma^{a_2}_{\lambda m}\Gamma^{a_1}_{k\mu}T^{\mu m}_{\bar{b}} - \Gamma^{a_1}_{km}\Gamma^{a_2}_{\lambda\mu}T^{m\mu}_{\bar{b}}}_{=0} \\ &= \Gamma^{a_1}_{\lambda m}\Gamma^{m}_{k\mu}T^{\mu a_2}_{\bar{b}} + \Gamma^{a_2}_{\lambda m}\Gamma^{m}_{k\mu}T^{a_1\mu}_{\bar{b}} - \Gamma^{a_1}_{km}\Gamma^{m}_{\lambda\mu}T^{\mu a_2}_{\bar{b}} - \Gamma^{a_2}_{km}\Gamma^{m}_{\lambda\mu}T^{a_1\mu}_{\bar{b}} \\ &= \left(\Gamma^{a_{j_p}}_{\lambda m}\Gamma^{m}_{k\mu} - \Gamma^{a_{j_p}}_{km}\Gamma^{m}_{\lambda\mu}\right)T^{\bar{a}[a_{j_p}\_\mu]}_{\bar{b}} \end{aligned}$$

$= \left(\Gamma^{a_{j_{p+1}}}_{\lambda m}\Gamma^m_{k\mu} - \Gamma^{a_{j_{p+1}}}_{km}\Gamma^m_{\lambda\mu}\right) T^{\bar{a}[a_{j_{p+1}}\_\mu]}_{\bar{b}}.$

Ce qui termine la preuve. Il reste à montrer que l'égalité suivante, qui correspond à G2, est vraie :

$$\mathrm{P}(q) : \Gamma^\mu_{\lambda b_{j_q}} \left(\Gamma^m_{kb_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} - \Gamma^\mu_{kb_{j_q}} \left(\Gamma^m_{\lambda b_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} = \left(\Gamma^m_{k\mu}\Gamma^\mu_{\lambda b_{i_q}} - \Gamma^m_{\lambda\mu}\Gamma^\mu_{kb_{i_q}}\right) T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}$$

Vérifions pour $\mathrm{P}(1)$:

$\Gamma^\mu_{\lambda b_1}\left(\Gamma^m_{kb_1}T^{\bar{a}}_m\right)_{[b_1\_\mu]} - \Gamma^\mu_{kb_1}\left(\Gamma^m_{\lambda b_1}T^{\bar{a}}_m\right)_{[b_1\_\mu]}$

$= \Gamma^\mu_{\lambda b_1}\left(\Gamma^m_{k\mu}T^{\bar{a}}_m\right) - \Gamma^\mu_{kb_1}\left(\Gamma^m_{\lambda\mu}T^{\bar{a}}_m\right)$

$= \left(\Gamma^\mu_{\lambda b_1}\Gamma^m_{k\mu} - \Gamma^\mu_{kb_1}\Gamma^m_{\lambda\mu}\right) T^{\bar{a}}_m.$

Donc $\mathrm{P}(1)$ est vérifiée. Supposons maintenant que $\mathrm{P}(q)$ est vraie et vérifions $\mathrm{P}(q+1)$:

$\Gamma^\mu_{\lambda b_{j_{q+1}}}\left(\Gamma^m_{kb_{i_{q+1}}} T^{\bar{a}}_{\bar{b}[b_{i_{q+1}}\_m]}\right)_{[b_{j_{q+1}}\_\mu]} - \Gamma^\mu_{kb_{j_{q+1}}}\left(\Gamma^m_{\lambda b_{i_{q+1}}} T^{\bar{a}}_{\bar{b}[b_{i_{q+1}}\_m]}\right)_{[b_{j_{q+1}}\_\mu]}$

$= \Gamma^\mu_{\lambda b_{j_q}}\left(\Gamma^m_{kb_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} + \Gamma^\mu_{\lambda b_{q+1}}\left(\Gamma^m_{kb_{q+1}} T^{\bar{a}}_{\bar{b}[b_{q+1}\_m]}\right)_{[b_{j_{q+1}}\_\mu]}$

$-\Gamma^\mu_{kb_{j_q}}\left(\Gamma^m_{\lambda b_{i_q}} T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}\right)_{[b_{j_q}\_\mu]} - \Gamma^\mu_{kb_{q+1}}\left(\Gamma^m_{\lambda b_{q+1}} T^{\bar{a}}_{\bar{b}[b_{q+1}\_m]}\right)_{[b_{q+1}\_\mu]}$

$= \left(\Gamma^m_{k\mu}\Gamma^\mu_{\lambda b_{i_q}} - \Gamma^m_{\lambda\mu}\Gamma^\mu_{kb_{i_q}}\right) T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]}$

$+\Gamma^\mu_{\lambda b_{q+1}}\left(\Gamma^m_{k\mu} T^{\bar{a}}_{\bar{b}[b_{q+1}\_m]}\right) - \Gamma^\mu_{kb_{q+1}}\left(\Gamma^m_{\lambda\mu} T^{\bar{a}}_{\bar{b}[b_{q+1}\_m]}\right)$

$= \left(\Gamma^m_{k\mu}\Gamma^\mu_{\lambda b_{i_q}} - \Gamma^m_{\lambda\mu}\Gamma^\mu_{kb_{i_q}}\right) T^{\bar{a}}_{\bar{b}[b_{i_q}\_m]} + \left(\Gamma^\mu_{\lambda b_{q+1}}\Gamma^m_{k\mu} - \Gamma^\mu_{kb_{q+1}}\Gamma^m_{\lambda\mu}\right) T^{\bar{a}}_{\bar{b}[b_{q+1}\_m]}$

$= \left(\Gamma^m_{k\mu}\Gamma^\mu_{\lambda b_{i_{q+1}}} - \Gamma^m_{\lambda\mu}\Gamma^\mu_{kb_{i_{q+1}}}\right) T^{\bar{a}}_{\bar{b}[b_{i_{q+1}}\_m]}.$

Ce qui termine la preuve. Substituant maintenant G1 et G2 dans le commutateur de la dérivée covariante, on trouve

$$\begin{aligned}
T^{\bar{a}}_{\bar{b};k\lambda} - T^{\bar{a}}_{\bar{b};\lambda k} &= \left(\Gamma^{a_{ip}}_{km,\lambda} - \Gamma^{a_{ip}}_{\lambda m,k} - C^\mu_{\lambda k}\Gamma^{a_{ip}}_{\mu m}\right) T^{\bar{a}[a_{i_p}_m]}_{\bar{b}} \\
&\quad + \left(\Gamma^m_{\lambda b_{i_q},k} - \Gamma^m_{kb_{i_q},\lambda} - C^\mu_{k\lambda}\Gamma^m_{\mu b_{i_q}}\right) T^{\bar{a}}_{\bar{b}[b_{i_q}_m]} \\
&\quad + C^m_{k\lambda}T^{\bar{a}}_{\bar{b},m} + \left(\Gamma^{a_{i_p}}_{\lambda m}\Gamma^m_{k\mu} - \Gamma^{a_{i_p}}_{km}\Gamma^m_{\lambda\mu}\right) T^{\bar{a}[a_{i_p}_\mu]}_{\bar{b}} \\
&\quad + \left(\Gamma^m_{k\mu}\Gamma^\mu_{\lambda b_{i_q}} - \Gamma^m_{\lambda\mu}\Gamma^\mu_{kb_{i_q}}\right) T^{\bar{a}}_{\bar{b}[b_{i_q}_m]} \\
&= \left(\Gamma^{a_{ip}}_{km,\lambda} - \Gamma^{a_{ip}}_{\lambda m,k} - C^\mu_{\lambda k}\Gamma^{a_{ip}}_{\mu m} + \Gamma^{a_{i_p}}_{\lambda m}\Gamma^m_{k\mu} - \Gamma^{a_{i_p}}_{km}\Gamma^m_{\lambda\mu}\right) T^{\bar{a}[a_{i_p}_m]}_{\bar{b}} \\
&\quad + \left(\Gamma^m_{\lambda b_{i_q},k} - \Gamma^m_{kb_{i_q},\lambda} - C^\mu_{k\lambda}\Gamma^m_{\mu b_{i_q}} + \Gamma^m_{k\mu}\Gamma^\mu_{\lambda b_{i_q}} - \Gamma^m_{\lambda\mu}\Gamma^\mu_{kb_{i_q}}\right) T^{\bar{a}}_{\bar{b}[b_{i_q}_m]} \\
&\quad + C^m_{k\lambda}T^{\bar{a}}_{\bar{b},m}.
\end{aligned}$$

Utilisant les expressions pour R, données par

$$\begin{aligned} R^{a_{i_p}}_{mk\lambda} &= \Gamma^{a_{i_p}}_{\lambda m,k} - \Gamma^{a_{i_p}}_{km,\lambda} - \Gamma^{a_{i_p}}_{lm} C^l_{k\lambda} + \Gamma^{a_{i_p}}_{kl} \Gamma^l_{\lambda m} - \Gamma^{a_{i_p}}_{\lambda l} \Gamma^l_{km}, \\ R^m_{b_{i_q} k\lambda} &= \Gamma^m_{\lambda b_{i_q},k} - \Gamma^m_{kb_{i_q},\lambda} - \Gamma^m_{\mu b_{i_q}} C^\mu_{k\lambda} + \Gamma^m_{k\mu} \Gamma^\mu_{\lambda b_{i_q}} - \Gamma^m_{\lambda\mu} \Gamma^\mu_{kb_{i_q}}. \end{aligned}$$

On trouve finalement que

$$T^{\bar{a}}_{\bar{b};k\lambda} - T^{\bar{a}}_{\bar{b};\lambda k} = -R^{a_{i_p}}_{mk\lambda} T^{\bar{a}[a_{i_p} \_ m]}_{\bar{b}} + R^m_{b_{i_q} k\lambda} T^{\bar{a}}_{\bar{b}[b_{i_q} \_ m]} + C^m_{k\lambda} T^{\bar{a}}_{\bar{b},m}. \clubsuit \qquad (**)$$

La preuve a été faite pour un torsion non-nulle ($C^m_{k\lambda} \neq 0$), mais le résultat demandé est trouvé si on la suppose nulle.