The Scalar Transport Equation of Coalescence Theory: Moments and Kernels

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ABSTRACT

This paper is concerned with complementing the known analytical studies of the pure coalescence equation. There is still a need for better analytical tools for the analysis of this problem even though high-speed computers have contributed much to the knowledge of this system. Specifically, when the detailed microphysics is incorporated into a large-scale, three-dimensional, moist, deep-convection model, it is currently impossible to solve the coalescence equation numerically for several size categories. Hence, there is a need for better analytical tools. In particular, we are concerned with the relationships between integral power moments of the size spectrum and the collection kernel, relationships between Friedlander's similarity solutions and the kernel, bounds for the size spectrum, and various power-moment inequalities. The results we obtained will allow us to make reasonable approximations for spectra which can, in turn, be used in large-scale convection models.

1. Introduction

The scalar transport equation being considered in this paper is concerned with the coalescence or coagulation of particles in a large, but finite, segment V of a homogeneous cloud of aerosol particles or water droplets. In this volume V there is no inflow or outflow of particles and no particle breakup or production; the only process which is acting is coalescence or coagulation. In addition, the physics of the coalescence process is independent of time; that is, the kernel in the scalar transport equation only depends upon the size or mass of the particles. Briefly, the physical problem can be stated as follows: initially, there is a "continuous" distribution of various sizes of particles throughout the volume V; then as time goes on, the particles collide and coalesce in such a manner that total mass is conserved. The scalar transport equation is a statement of this conservation law.

Several fine papers have been written in recent years concerning various analytical properties of the scalar transport equation. Some of the topics studied have been the existence, uniqueness, positivity, boundedness and continuity of solutions, the continuity of solutions with reference to variations in the initial size spectra and collection kernels, and the existence of similarity solutions. These results have greatly added to the knowledge of the characteristics of the transport equation, and have proven useful in extending the list of known exact solutions for the system and in checking new approximate solution techniques. Through the application of these properties, certain candidates for the initial spectra and collection kernels have been eliminated.

Even though high-speed digital computers have been used to numerically solve the transport equation (Berry, 1967), there is still a great need to study the properties of the solutions of this system. Knowledge of these analytical properties can be used in the assessment of the sensitivity of the evolving size spectrum to the forms of the initial spectrum and the collection kernel, to evaluate the proper type of approximations for "physical" type kernels, and to determine the potential accuracy of finite-difference schemes. All of these questions are important and cannot be answered to any great degree from the study of the numerical solutions because of the inaccuracies due to the large range over which the particle size and the size spectrum vary. The use of logarithms by Berry (1967) does not completely eliminate this problem with numerical solutions, especially in the "tail" of the spectrum.

In this author's opinion there is even a more important reason for seeking new analytical properties of the solutions for the coalescence equation. This new reason is a result of the following project being conducted at the National Center for Atmospheric Research (NCAR). We are currently constructing a three-dimensional, moist, deep-convection model which will initially consider the following dependent variables: three velocity components, pressure, density, absolute temperature, and the mixing ratios of water vapor, cloud water, cloud ice, rain and hail. The domain of integration will be 50 km by 50 km by 20 km with a mesh size of 500 m in each direction. For our computer

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system, the CDC 7600, it will be impossible to consider 20–40 particle sizes for each of the water categories (cloud water, cloud ice, rain and hail) and then solve coalescence-type equations to determine how these particles combine. However, we will want a better microphysical formulation than mixing ratios and mean particle sizes as our cloud model evolves. Possibly a better formulation may be obtained, without an excessive increase in the number of dependent variables, through the use of the method of moments [see Stieltjes (1894, 1895) and Golovin (1963a, b)]. Another possibility may be to approximate the various size spectra by using upper and lower bounds for the spectra, or by first approximating physical-type kernels by a sum of simpler kernels and then obtaining the exact solutions corresponding to the simpler kernels. Hence, the purpose of this paper is to determine new properties of the coalescence equation which will be useful in developing approximate formulations for the microphysics in large-scale convective models.

The first results which we obtain are the integral power moments corresponding to the collection kernel formed by the linear combination of the constant, sum and product kernels. From these formulas, we conclude that the product kernel, or any kernel containing the product kernel, is indeed a special case.

Using various algebraic and integral inequalities we derive important power-moment inequalities, some of which are independent of and others which are dependent on the collection kernel. The form of the kernel not only strongly dictates the character of the power moments but is also very influential in the determination of the solution techniques for the transport equation. An important solution technique in the general field of mechanics and physics is the search for similarity solutions. We show that the existence of a certain type of similarity solution is greatly restricted by the form of the kernel.

Finally, we consider the implications of the famous "moment problem" of Stieltjes (1894, 1895) on the type of solution and the method of solving the transport equation. Assuming that the solutions of the coalescence equation satisfy the hypotheses of the moment problem, we are then able to consider the method of moments for obtaining solutions and bounds for these solutions.

2. The scalar transport equation

The continuous form of the scalar transport equation was probably first derived by Müller (1928). It is given by

$$\frac{\partial n(v,t)}{\partial t} = -n(v,t)\int_0^\infty K(v,u)n(u,t)du + \frac{1}{2}\int_0^v K(v-u,\,u)n(v-u,\,t)n(u,t)du, \quad (2.1)$$

where u and v are the volumes of spherical particles, t is time, $n(v,t)$ the time-varying size spectrum or particle size density function, and $K(u,v)$ the collection kernel. The first integral in (2.1) accounts for the disappearance of the v particles due to their coalescence with u particles, while the second integral accounts for the production of v particles due to the coalescence of u and (v,u) particles. If an initial spectrum

$$n(v,0) = n(v) \quad (2.2)$$

is specified along with the collection kernel $K(u,v)$, then the solution of (2.1) and (2.2) gives the time evolution of the initial spectrum under the particular coalescence process described by $K(u,v)$.

In 1957, Melzak proved that if $n(v)$ and $K(u,v)$ possessed certain boundedness, continuity and positivity properties, then there exists a unique, nonnegative, continuous and bounded solution of (2.1) and (2.2). McLeod (1964) extended Melzak's results to include a certain class of unbounded kernels, namely $K=Auv$, where A is a positive constant. The theory of Melzak and McLeod concerning the existence and uniqueness of solutions of (2.1) and (2.2) is the only theory of this kind known to the author of this paper. Hence, throughout the following discussion we will either assume that the above theory is valid or we will invoke the following hypothesis: "assuming that $n(v,t)$ exists and is unique, then 'such and such' a result will follow." Other hypotheses which are assumed in this paper are

$$n(v,0) = n(v) \geqslant 0, \qquad 0 \leqslant v \leqslant \infty, \quad (2.3)$$

$$K(u,v) = K(v,u) \geqslant 0, \quad 0 \leqslant u,\, v \leqslant \infty, \quad (2.4)$$

where both $n(v)$ and $K(u,v)$ are continuous for $u,v \geqslant 0$; finally, $n(v)$ is assumed to be bounded, while $K(u,v)$ may or may not be bounded.

In our study it is convenient to nondimensionalize the system. Following the work of Scott (1968), the nondimensional terms are defined as follows:

$$\left.\begin{aligned} u &= v_0 y \\ v &= v_0 x \\ \tau &= EN_0 t \\ K(u,v) &= E\alpha(y,x) \\ v_0 n(v,t) &= N_0 f(x,\tau) \\ v_0 n(v) &= N_0 f(x,0) = N_0 f_0(x) \end{aligned}\right\}, \quad (2.5)$$

where N_0 is the initial total number density, v_0 the initial mean volume of the particles, x and y are nondimensional volumes, $\alpha(x,y)$ the nondimensional collection kernel, τ the dimensionless time, $f(x,\tau)$ the dimensionless size spectrum, and E a normalizing factor with dimensions of volume per unit time. Substituting

(2.5) into (2.1) gives

$$\frac{\partial f(x,\tau)}{\partial\tau}=-f(x,\tau)\int_0^\infty \alpha(x,y)f(y,\tau)dy+\frac{1}{2}\int_0^x \alpha(x-y,\,y)f(x-y,\,\tau)f(y,\tau)dy,\quad (2.6)$$

where $f(x,0)=f_0(x)$ and $\alpha(x,y)=\alpha(y,x)\geqslant 0$. The ν-integral power moment of $f(x,\tau)$ is defined by

$$M_\nu(\tau)=\int_0^\infty x^\nu f(x,\tau)dx,\quad (2.7)$$

where $\nu\geqslant 0$. Scott (1968) and Thompson (1968) have shown that

$$M_0(0)=1\quad\text{and}\quad M_1(\tau)=1.\quad (2.8)$$

The last equation in (2.8) is an expression for the conservation of mass within a unit volume of cloud.

3. Power moments for $\alpha(x,y)=A+B(x+y)+Cxy$

For any kernel $\alpha(x,y)$ we can derive a set of ordinary integro-differential equations for the moments M_ν defined in (2.7) [e.g., Thompson (1968)]. If we multiply (2.6) by x^ν, $\nu\geqslant 0$, and integrate over the interval $0\leqslant x\leqslant\infty$, we get

$$\frac{dM_\nu(\tau)}{d\tau}=\frac{1}{2}\int_0^\infty\int_0^\infty [(x+y)^\nu-x^\nu-y^\nu]\alpha(x,y)\times f(x,\tau)f(y,\tau)dxdy.\quad (3.1)$$

For the remainder of this section let us consider only integer ν's; that is, let $\nu=N=0,1,2,3,\cdots$. For $N=0$, (3.1) reduces to

$$\frac{dM_0(\tau)}{d\tau}=-\frac{1}{2}\int_0^\infty\int_0^\infty \alpha(x,y)f(x,\tau)f(y,\tau)dxdy.\quad (3.2)$$

For $N=2,3,4,\cdots$, we have

$$\frac{dM_N(\tau)}{d\tau}=\frac{1}{2}\sum_{i=1}^{N-1}\binom{N}{i}\int_0^\infty\int_0^\infty x^iy^{N-i}\alpha(x,y)\times f(x,\tau)f(y,\tau)dxdy,\quad (3.3)$$

where $\binom{N}{i}$ are binomial coefficients. For $N=1$, we obtain the result given in (2.8).

If $\alpha(x,y)$ is a polynomial containing x^2 and y^2 terms, or terms of higher degree, then the right side of (3.3) contains higher order moments than order N. Therefore, the system of equations does not "close"; that is, we cannot solve for the Nth moment without knowing the $(N+1)$st, or higher order moments first. In order to eliminate this "closure problem," we will consider the kernel

$$\alpha(x,y)=A+B(x+y)+Cxy,\quad (3.4)$$

where A, B and C are non-negative constants. Also, we assume that the constant E in (2.5) has been chosen such that $\max[A,B,C]=1$.

Combining (3.2) and (3.4) gives

$$\frac{dM_0(\tau)}{d\tau}=-\tfrac{1}{2}[AM_0^2(\tau)+2BM_0(\tau)+C],\quad (3.5)$$

where $M_0(0)=1$. From (3.3) and (3.4) we have

$$\frac{dM_N(\tau)}{d\tau}=\frac{1}{2}\sum_{i=1}^{N-1}\binom{N}{i}[AM_i(\tau)M_{N-i}(\tau)+BM_{i+1}(\tau)M_{N-i}(\tau)+BM_i(\tau)M_{N+1-i}(\tau)+CM_{i+1}(\tau)M_{N+1-i}(\tau)],\quad (3.6)$$

where $N=2,3,4,\cdots$, and where

$$M_N(0)=\int_0^\infty x^Nf_0(x)dx.\quad (3.7)$$

Eq. (3.5) can be easily solved for any A, B, and C; and (3.6) and (3.7) can be solved inductively for all $N=2,3,\cdots$.

The set of solutions for (3.5) is given by

$$M_0(\tau)=\frac{2-(B+C)\tau}{2+(A+B)\tau},\quad B^2=AC;\quad (3.8a)$$

$$M_0(\tau)=\frac{D^{\frac{1}{2}}-(B+C)\tan(\tau D^{\frac{1}{2}}/2)}{D^{\frac{1}{2}}+(A+B)\tan(\tau D^{\frac{1}{2}}/2)},\quad D=AC-B^2>0,\quad (3.8b)$$

$$M_0(\tau)=\frac{F^{\frac{1}{2}}[1+\exp(-F^{\frac{1}{2}}\tau)]-(B+C)[1-\exp(-F^{\frac{1}{2}}\tau)]}{F^{\frac{1}{2}}[1+\exp(-F^{\frac{1}{2}}\tau)+(A+B)[1-\exp(-F^{\frac{1}{2}}\tau)]},\quad F=B^2-AC>0.\quad (3.8c)$$

From these equations we can easily show that whenever $C=0$, $M_0(\tau)>0$ for $\tau\geqslant 0$ and $M_0(\tau)\to 0^+$ as $\tau\to+\infty$ [see Drake (1971) for the detailed mathematics]. Similarly, if $C\neq 0$, then there exists a finite value of τ, say $\tau_{\max}$, such that $M_0(\tau_{\max})=0$. That is, the total number density approaches zero in finite time, or coalescence stops at $\tau=\tau_{\max}$, unless there are source or breakup terms in (2.6). It can be shown that $\tau_{\max}\leqslant 2$ whenever $C=1$; and as $C\to 0^+$, $\tau_{\max}\to+\infty$. Therefore, whenever C is significantly close to unity, the coalescence process is very rapid and $M_0(\tau)$ becomes zero in a relatively short time. This indicates that an admissible kernel (from the physical point of view) should not increase as rapidly as $\alpha=xy$.

Suppose $N=2$ in (3.6); then we have

$$\frac{dM_2(\tau)}{d\tau}=A+2BM_2(\tau)+CM_2^2(\tau), \qquad (3.9)$$

where $M_2(0)=M_{20}$. The set of solutions for (3.9) is given by

$$M_2(\tau)=\frac{M_{20}+[A+BM_{20}]\tau}{1-[B+CM_{20}]\tau}, \quad AC=B^2, \qquad (3.10a)$$

$$M_2(\tau)=\frac{D^{\frac{1}{2}}M_{20}+[A+BM_{20}]\tan(D^{\frac{1}{2}}\tau)}{D^{\frac{1}{2}}-[B+CM_{20}]\tan(D^{\frac{1}{2}}\tau)},$$
$$D=AC-B^2>0, \qquad (3.10b)$$

$$M_2(\tau)$$
$$=\frac{F^{\frac{1}{2}}M_{20}[1+\exp(2F^{\frac{1}{2}}\tau)]-(A+BM_{20})[1-\exp(2F^{\frac{1}{2}}\tau)]}{F^{\frac{1}{2}}[1+\exp(2F^{\frac{1}{2}}\tau)]+(B+CM_{20})[1-\exp(2F^{\frac{1}{2}}\tau)]},$$
$$F=B^2-AC>0. \qquad (3.10c)$$

Whenever $C=0$ (Drake, 1971), $M_2(\tau)>0$ for $\tau\geqslant 0$ and $M_2(\tau)\rightarrow+\infty$ as $\tau\rightarrow+\infty$. If $C\neq 0$, there is always a finite value of τ, say τ_∞, at which $M_2(\tau)\rightarrow+\infty$. We can also show that $\tau_\infty\leqslant\frac{1}{2}\tau_{\max}$. Hence, τ_∞ is probably the upper bound on the τ's for realistic size spectra. Whenever C is close to unity, τ_∞ is close to unity; and as $C\rightarrow 0^+$, $\tau_\infty\rightarrow+\infty$. Therefore, we have another indication of the poor choice of $\alpha=xy$ as a candidate for a "physical kernel."

Another observation concerning $M_2(\tau)$ is that it is dependent upon the initial spectrum through the quantity M_{20}. This is not necessarily true of the moments $M_0(\tau)$ and $M_1(\tau)$. The moment $M_1(\tau)$ is independent of both $f_0(x)$ and $\alpha(x,y)$ over its domain of definition; however, the domain of definition is kernel-dependent. The only known kernel for which $M_0(\tau)$ is independent of the initial spectrum $f_0(x)$ is that given in (3.4).

In Drake (1971), we solved the system given in (3.6) for all N for $\alpha=1$, $x+y$, and xy. From these results we conjecture that as the degree of homogeneity of the kernel $\alpha(x,y)$ increases, the dependence of the moments [and hence the spectrum $f(x,\tau)$ itself] upon the initial spectrum becomes stronger. For example, when $\alpha=1$, the power moments are polynomials in τ with a leading coefficient independent of $f_0(x)$. However, when $\alpha=xy$, the power moments are rational expressions in τ and contain initial values of M_N in all terms. In addition, the moments become infinite at $\tau=M_{20}^{-1}$. To get some idea of the range of values for M_{20}, we consider the following initial spectra. If $f_0(x)=e^{-x}$, then $M_{20}=2$ and if $f_0(x)=\delta(x-1)$, the Dirac delta function, then $M_{20}=1$. Hence $M_2(\tau)$ for $\alpha=xy$ tends to infinity as τ approaches some number between 1/2 and 1, while $M_0(\tau)$ is zero at $\tau=2$. This last point will be important when we consider the radius of convergence of the power series solution for $\alpha=xy$ in the next section.

4. Exact solutions corresponding to $\alpha(x,y)=xy$

McLeod (1964) proved a series of existence and uniqueness theorems for (2.6) with $\alpha(x,y)=xy$ and for an initial spectrum $f_0(x)$. The most important conclusion from McLeod's work is the fact that we can only guarantee the existence of unique solutions over a finite time interval $0\leqslant\tau\leqslant\delta$, where δ is determined by the second power moment. Hence, from Section 3, we know that δ must be a fixed number $<M_{20}^{-1}$. In fact, if $\tau>\delta$, then we no longer can satisfy the conservation of mass property, $M_1(\tau)=1$. McLeod's theorems were not proved for the kernel in (3.4), but we conjecture that the τ's obtained from (3.10) are probably the upper bounds of the intervals of existence for the solutions corresponding to the kernel in (3.4). Therefore, if C is close to unity, the kernel and, hence, the spectra are not very realistic for physical problems. These results also indicate the danger of expanding a bounded kernel or a kernel increasing slower than xy in terms of a truncated power series and then solving the resulting system; an example of this procedure was proposed by Thompson (1968).

Scott (1968) considered the kernel $\alpha=xy$ for general initial spectra. With the exception of a special generalized spectrum, Scott studied spectra which satisfy the following:

$$\left.\begin{array}{l}\int_0^\infty f_0(x)dx=\int_0^\infty xf_0(x)dx=1\\ f_0(x)\geqslant 0,\quad x\geqslant 0\\ f_0(x)=0,\quad x<0\\ f_0(x)\not\equiv 0,\quad \text{for } x\geqslant 1,\quad \text{or for } x\leqslant 1\\ L[f_0(x)]=G(s)\text{ exists}\end{array}\right\}, \qquad (4.1)$$

where $G(s)$ is the Laplace transform of $f_0(x)$. By using Laplace transforms Scott solved (2.6) for $\alpha=xy$ and the general $f_0(x)$ in (4.1). The result is

$$f(x,\tau)=\frac{e^{-\tau x}}{x}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(x\tau)^k}{(k+1)!}L^{-1}\left\{\left[\frac{dG(s)}{ds}\right]^{k+1}\right\}, \qquad (4.2)$$

where $L^{-1}[\ ]$ is the inverse Laplace transform.

Scott considered a special family of $f_0(x)$'s, namely, the gamma distributions given by

$$f_0(x)=\frac{(\nu+1)^{\nu+1}x^\nu e^{-(\nu+1)x}}{\Gamma(\nu+1)}, \quad \nu\geqslant 0. \qquad (4.3)$$

Suppose we now substitute (4.3) into (4.2) where $\nu=N=0, 1, 2, \cdots$. Then, if we multiply the resulting series by x^L, $L=0, 1, 2, \cdots$, and integrate with respect to x from 0 to ∞, the resulting series is

$$M_L(\tau)=\sum_{n=0}^{\infty}\frac{\tau^n(N+1)^{(N+2)(n+1)}[N(n+1)+L+3n]!}{(n+1)![N(n+1)+2n+1]![N+1+\tau]^{(N+3)n+L+N+1}}. \quad (4.4)$$

If A_n is the nth term in the series in (4.4), then the ratio, in the "ratio test," is given by

$$R(\tau)=\lim_{n\to\infty}\frac{A_{n+1}}{A_n}=\left(\frac{N+1}{N+2}\right)^{N+2}\left(\frac{N+3}{\tau+N+1}\right)^{N+3}\tau. \quad (4.5)$$

The maximum value of $R(\tau)$ is unity and it occurs at

$$\tau_{cr}=\frac{N+1}{N+2}=\frac{1}{M_2(0)}. \quad (4.6)$$

Hence, for the uniform convergence of (4.4) for $L=0,1,2,\cdots$, τ must be in the interval $0\leqslant\tau\leqslant\delta<1/M_2(0)$. This result is consistent with McLeod's theorems and with the results given in Section 3.

Scott derived an asymptotic formula for the series given in (4.2). For (4.3), where $\nu=N=0,1,2,\cdots$, this expansion is given by

$$f(x,\tau)\sim\tau^{-(2N+5)/(2N+6)}\left[\frac{1}{2\pi x^5}\left(\frac{N+1}{N+3}\right)\left(\frac{N+2}{N+1}\right)^{1/(N+3)}\right]^{\frac{1}{2}}\times\exp[-(N+1)F(\tau)], \quad (4.7)$$

where

$$F(\tau)=1+\frac{\tau}{N+1}-\left(\frac{N+3}{N+1}\right)\left(\frac{N+1}{N+2}\right)^{(N+2)/(N+3)}\tau^{1/(N+3)}. \quad (4.8)$$

Since the first τ factor on the right side of (4.7) is not very influential, let us consider only the term $F(\tau)$. The minimum value of $F(\tau)$ is zero and occurs at $\tau_{cr}=M_2^{-1}(0)$. Hence, we have the same result as we obtained from the series. That is, (4.7) is only valid for $0\leqslant\tau\leqslant M_2^{-1}(0)$.

This analysis of both the series (4.4) and the asymptotic form (4.7), the theory of McLeod, and the moment relationships in Section 3 explain why the graphical results given by Scott (1968) for $\alpha=xy$ do not exhibit the conservation of mass law, but, in fact, indicate that mass is destroyed after a certain time, namely, when $\tau>\tau_{cr}=M_2^{-1}(0)$. That mass is destroyed by Scott's results is most easily seen from the asymptotic form since $F(\tau)$ increases for $\tau>\tau_{cr}$.

In order to show the physical restrictions placed on realistic cloud processes by the xy kernel, let us consider the example given by Scott. Here the value of $K(u,v)$ in (2.1) is assumed to be 1.80×10^{-4} cm^3 sec^{-1} when $u=v_0=4.189\times10^{-9}$ cm^3 and $v=1.131\times10^{-7}$ cm^3. The total water content N_0v_0 is assumed to be 10^{-6} gm cm^{-3}. In Table 1 we have the values of the normalized total number density $M_0(\tau)$, dimensionless time τ, and dimensional time t in seconds. If $M_2(0)=2$, then the series solution in (4.2) is only valid up to 315 sec and the number density has been reduced by only 25%. For $M_2(0)=1$, the corresponding values are 629 sec and 50%. Hence, if we seek size spectra which possess Laplace transforms (and this seems reasonable), it appears that the xy kernel is highly restrictive physically speaking.

5. Some general properties of power moments

In Section 3 we have seen the manner in which $M_0(\tau)$ and $M_2(\tau)$ vary with the form of the collection kernel. In this section we consider more general properties of the moments $M_\nu(\tau)$. Some of these properties are only ν-dependent while other properties also depend upon the collection kernel. The results obtained in this section will be useful in the analysis of (2.6) by the "method of moments." The details of the mathematics for this work is given in Drake (1971).

From Beckenbach and Bellman (1965) we have

$$(x+y)^\nu-x^\nu-y^\nu\geqslant0, \quad (5.1a)$$

if $\nu\geqslant1$, $x\geqslant0$, $y\geqslant0$, and

$$(x+y)^\nu-x^\nu-y^\nu\leqslant0, \quad (5.1b)$$

if $0\leqslant\nu\leqslant1$, $x\geqslant0$, $y\geqslant0$. Combining (3.1) and (5.1) and assuming that α and f are non-negative proves that $M_\nu(\tau)$ monotonically increases with increasing τ for $\nu>1$ and $M_\nu(\tau)$ monotonically decreases for $0\leqslant\nu<1$.

Also, from Beckenback and Bellman, we have

$$x^\nu\geqslant\nu x+1-\nu,\quad \nu\geqslant1,\quad x\geqslant0, \quad (5.2a)$$

$$x^\nu\leqslant\nu x+1-\nu,\quad 0<\nu\leqslant1,\quad x\geqslant0. \quad (5.2b)$$

TABLE 1. Relationships between $M_0(\tau)$, τ and t for $\alpha=xy$.

$M_0(\tau)$	τ	t (sec)
1	0	0
0.95	0.1	63
0.90	0.2	126
0.85	0.3	189
0.80	0.4	252
0.75	0.5	315
0.70	0.6	377
0.60	0.8	503
0.50	1.0	629
0.40	1.2	755
0.30	1.4	881
0.20	1.6	1070
0.10	1.8	1131
0.05	1.9	1192
0.01	1.98	1245
0.001	1.998	1255

Combining (2.7) and (5.2) gives the moment relationships

$$M_\nu(\tau)\geqslant\nu+(1-\nu)M_0(\tau),\quad \nu\geqslant 1, \tag{5.3a}$$

$$M_\nu(\tau)\leqslant\nu+(1-\nu)M_0(\tau),\quad 0\leqslant\nu\leqslant 1. \tag{5.3b}$$

Since $M_0(0)=1$ and $M_0(\tau)$ monotonically decreases with τ, then for all $\tau\geqslant 0$, we have

$$\left.\begin{array}{ll} M_\nu(\tau)\geqslant 1, & \nu\geqslant 1 \\ M_\nu(\tau)\leqslant 1, & 0\leqslant\nu\leqslant 1\end{array}\right\}. \tag{5.3c}$$

From Feller (1966), we have

$$\left.\begin{array}{l} x^a\leqslant x^b+1 \\ x\geqslant 0,\quad 0<a\leqslant b\end{array}\right\}. \tag{5.4a}$$

Substituting (5.4a) into (2.7) gives

$$M_a(\tau)\leqslant M_b(\tau)+M_0(\tau) \tag{5.4b}$$

for $0\leqslant a\leqslant b$. Over part of the range for a and b, the inequality in (5.4b) can be improved. That is, using the results

$$\left.\begin{array}{lll} x^\nu-x\geqslant 0, & \nu\geqslant 1, & x\geqslant 1 \\ x^\nu-x\leqslant 0, & \nu\geqslant 1, & 0<x\leqslant 1\end{array}\right\}, \tag{5.5a}$$

we can show that

$$1\leqslant M_a(\tau)\leqslant M_b(\tau),\quad 1\leqslant a\leqslant b. \tag{5.5b}$$

Using Schwarz's integral inequality (von Mises, 1964)

$$\left[\int f(x)g(x)p(x)dx\right]^2 \leqslant \int f^2(x)p(x)dx\int g^2(x)p(x)dx, \tag{5.6a}$$

where f, g and p are non-negative, we can prove that

$$M_{a+b}^2(\tau)\leqslant M_{2a}(\tau)M_{2b}(\tau),\quad 0\leqslant a\leqslant b, \tag{5.6b}$$

if $f(x)=x^a$, $g(x)=x^b$ and $p(x)=f(x,\tau)$. The integration limits in (5.6a) are assumed to be from 0 to ∞. Through the use of mathematical induction, three interesting special cases of (5.6b) are obtained. They are

$$M_{qa}(\tau)\leqslant M_{pa}^{q/p}(\tau), \tag{5.7a}$$

where $0<a<\infty$, $q\leqslant p$, and p and q are positive integers; and

$$\left.\begin{array}{l} M_{1/(2^n)}(\tau)\leqslant[M_0(\tau)]^{(2^n-1)/2^n} \\ M_{(2^n-1)/2^n}(\tau)\leqslant M_0^{1/(2^n)}(\tau)\end{array}\right\}, \tag{5.7b}$$

where n is a positive integer.

Inequality (5.6b) can be generalized for integer indices by using double integrals. If

$$P(x,y)\leqslant Q(x,y),\quad 0\leqslant x,y\leqslant\infty, \tag{5.8a}$$

where a_{ij} and b_{ij} are constants, and

$$P(x,y)=\sum_{i=0}^{N}\sum_{j=0}^{N-i}a_{ij}x^iy^j, \tag{5.8b}$$

$$Q(x,y)=\sum_{i=0}^{M}\sum_{j=0}^{M-i}b_{ij}x^iy^j, \tag{5.8c}$$

then

$$\int_0^\infty\int_0^\infty[P(x,y)-Q(x,y)]f(x,\tau)f(y,\tau)dxdy\leqslant 0, \tag{5.9a}$$

or

$$\sum_{i=0}^{N}\sum_{j=0}^{N-i}a_{ij}M_i(\tau)M_j(\tau)\leqslant\sum_{i=0}^{M}\sum_{j=0}^{M-i}b_{ij}M_i(\tau)M_j(\tau). \tag{5.9b}$$

Now let us consider the kernel given by

$$\alpha(x,y)=(1/2)(x^{2p}+y^{2p}),\quad p\geqslant 1. \tag{5.10a}$$

From the quadratic inequality we have

$$\alpha(x,y)=1/2(x^{2p}+y^{2p})\geqslant x^py^p. \tag{5.10b}$$

Combining (5.10b), (5.5b), (3.2) and (3.3) gives

$$\frac{dM_0(\tau)}{d\tau}\leqslant-\tfrac{1}{2}M_0^2(\tau)\leqslant-\tfrac{1}{2}, \tag{5.11a}$$

$$\frac{dM_2(\tau)}{d\tau}\geqslant M_2^2(\tau). \tag{5.11b}$$

Solving these differential inequalities results in the relationships

$$M_0(\tau)\leqslant 1-\tau/2, \tag{5.12a}$$

$$M_2(\tau)\geqslant\frac{M_{20}}{1-M_{20}\tau}. \tag{5.12b}$$

The inequalities in (5.12) prove that the moment $M_0(\tau)$ corresponding to (5.10a) becomes zero as fast or faster than $M_0(\tau)$ for $\alpha=xy$, and the moment $M_2(\tau)$ approaches infinity as fast or faster than the xy-kernel case. Hence, we have another indication of the possible non-applicability for physical problems of kernels increasing as fast or faster than xy.

In fact, if a kernel corresponding to some physical process or an approximation to a physical kernel increases as fast or faster than xy for large values of x and y, then we should be highly suspicious of the derivation of the kernel or of the approximating technique being used. In searching the literature we see that this has not yet been the case for kernels describing physical phenomena. For example, the Brownian kernel is homogeneous of degree zero in x and y (Smoluchowski, 1917); the kernel for coalescence in a laminar shear flow is homogeneous of degree 1 (Smoluchowski, 1917); for coalescence in a turbulent diffusion field the degree is 1 (Saffman and Turner, 1956); for turbulent coagulation under an inertial mechanism the degree is 4/3 (Levich, 1962); and for gravitational coalescence Berry (1967) used a formula of degree 1, Golovin (1963a) a formula of degree 5/6, and Enukashvili (1964a) 4/3. Hence, all of these results are of degree <2 as in the case of $\alpha=xy$.

6. Similarity solutions vs kernels

The theory of "self-preserving spectra" was introduced by Friedlander (1961) to help explain experimentally observed regularities in certain particle size distributions of atmospheric aerosols. In Swift and Friedlander (1964) the theory was developed and tested experimentally. Friedlander and Wang (1966) and Wang and Friedlander (1967) refined and extended the theory. The search for a self-preserving spectrum, as outlined in the above papers, is really a search for a similarity solution of (2.6) which satisfies (2.8). A similarity solution, if it exists, is dependent upon the collection kernel and the initial spectrum; however, it is hoped that as $\tau\rightarrow\infty$, the actual spectrum approaches the self-preserving spectrum for any $f_0(x)$. This may be the case for the constant kernel or for a nonconstant, homogeneous kernel of degree 0 (see the concluding remarks in Section 3). However, for $\alpha=x+y$ and $\alpha=xy$, this is probably not true.

Friedlander and Wang considered a homogeneous kernel of degree μ, namely,

$$\alpha(px,py)=p^{\mu}\alpha(x,y). \tag{6.1}$$

They also derived the similarity transformation

$$f(x,\tau)=M_0^2(\tau)\psi(\mu), \tag{6.2}$$

$$\eta=M_0(\tau)x. \tag{6.3}$$

Combining (2.6), (6.1), (6.2) and (6.3) results in the ordinary integro-differential equation for ψ given by

$$\left[\eta\frac{d\psi}{d\eta}+2\psi\right]\left[\int_0^\infty\int_0^\infty \alpha(\eta,\xi)\psi(\eta)\psi(\xi)\right]d\eta d\xi$$
$$=2\int_0^\infty \alpha(\eta,\xi)\psi(\eta)\psi(\xi)d\xi$$
$$-\int_0^\eta \alpha(\eta-\xi,\,\xi)\psi(\eta-\xi)\psi(\xi)d\xi, \tag{6.4}$$

where

$$\int_0^\infty \eta\psi(\eta)d\eta=\int_0^\infty \psi(\eta)d\eta=1. \tag{6.5}$$

For $\alpha(\eta,\xi)=1$, (6.4) reduces to (Friedlander and Wang, 1966)

$$\eta\frac{d\psi}{d\eta}=-\int_0^\eta \psi(\eta-\xi)\psi(\xi)d\xi. \tag{6.6}$$

Through the application of Laplace transforms, the solution of (6.5) and (6.6) is given by

$$\psi(\eta)=e^{-\eta}, \tag{6.7}$$

which corresponds to the initial spectrum $f_0(x)=e^{-x}$. These authors have also obtained some numerical and asymptotic results for $\psi(\eta)$ for the Brownian kernel. It appears that the "self-preserving spectrum" for the Brownian kernel, which is homogeneous of degree 0, is that given in (6.7). In addition, several sets of aerosol data also tend to lie along the spectrum given by (6.7). The reason for these last two results and the reason that $f_0(x)$ is not too influential for large τ's may be explained by the moment results obtained in Section 3. That is, all integer power moments for the constant kernel are polynomials in τ whose leading coefficient is independent of $f_0(x)$. This result may also "carry over" to nonconstant, homogeneous kernels of degree 0.

Suppose we now seek a solution of (6.4) and (6.5) for the sum kernel, $\alpha=\xi+\eta$, and further, suppose that this solution possesses a Laplace transform in the ordinary sense. Eq. (6.4) reduces to

$$\eta\frac{d\psi}{d\eta}=(\eta-1)\psi-\tfrac{1}{2}\eta\int_0^\eta \psi(\eta-\xi)\psi(\xi)d\xi. \tag{6.8}$$

If the Laplace transform of $\psi(\eta)$ is defined by

$$L(\psi)=\Phi(s), \tag{6.9}$$

then the transform of (6.8) is given by

$$(\Phi-1+s)\Phi'=0. \tag{6.10}$$

The solutions of the transformed equation are

$$\left.\begin{array}{l}\Phi_1=\text{constant}\\ \Phi_2=1-s\end{array}\right\}. \tag{6.11}$$

Similarly, for the product kernel, $\alpha=\xi\eta$, (6.4) reduces to

$$\eta\frac{d\psi}{d\eta}=2(\eta-1)\psi-\int_0^\eta (\eta-\xi)\xi\psi(\eta-\xi)\psi(\xi)d\xi. \tag{6.12}$$

The Laplace transform of (6.12) is given by

$$(\Phi')^2+(2-s)\Phi'+\Phi=0, \tag{6.13}$$

the solutions of which are

$$\left.\begin{array}{l}\Phi_1=A(s-2)-A^2\\ \Phi_2=\tfrac{1}{4}(s-2)^2\end{array}\right\}, \tag{6.14}$$

where A is an arbitrary constant. While the inverse Laplace transformations of (6.11) and (6.14) are not defined in the ordinary sense, they do give generalized functions. Hence, we conclude that the sum and product kernels do not possess similarity solutions of the form given in (6.2) and (6.3), and do not have Laplace transforms in the ordinary sense. From these results and from the moment inequalities in the previous section, we conjecture that homogeneous kernels of degree $\geqslant 1$ do not possess similarity solutions of the type specified in the previous statement. We feel this is true even though Friedlander and Wang obtained some numerical and asymptotic results for the "laminar shear" kernel.

7. The method of moments

Golovin (1963b) and Enukashvili (1964a, b) have used the method of moments to obtain approximations for evolving spectra of coalescing cloud droplets in a rising air stream. The outline of this method for the system in (2.6) is as follows: Let us suppose we can develop $f(x,\tau)$ in a series

$$f(x,\tau)=\sum_{i=0}^{\infty} C_i(\tau)\psi_i(x)w(x), \qquad (7.1)$$

where ψ_i is a set of orthogonal polynomials with respect to the weight function $w(x)$. Assuming that (7.1) is valid, the rate of convergence of the series is strongly dependent upon the unknown weight functions. If one studies the forms of the exact solutions given in Scott (1968) for the constant, sum and product kernels, it is seen that the weight function $w(x)$ is, in turn, highly dependent upon the kernel of the system. Hence, choices for $w(x)$ could be based upon the known exact solutions. In fact, the most rapid convergence will be achieved when the quantity $w(x)$ is close to the unknown distribution $f(x,\tau)$. Hence, it is desirable to find a "good approximation" to $f(x,\tau)$ and then use this approximation for choosing $w(x)$.

If (2.6) is multiplied by x^N and if $f(x,\tau)$ is replaced by (7.1), then integrating with respect to x from 0 to ∞ gives an infinite set of equations in $M_N(\tau)$ and $C_i(\tau)$. Because of the orthogonality of the polynomials ψ_i, the amplitude functions $C_i(\tau)$ can be written in terms of the power moments. Orthogonality also guarantees that the resulting infinite system for the $M_N(\tau)$'s can be solved inductively. In addition, if the series in (7.1) is truncated at $i=I$, then the $C_i(\tau)$'s are all zero for $i>I$. That is, there is a closed system for the determination of the moments $M_N(\tau)$; this system is obviously different from that given in (3.3). This new system for $M_N(\tau)$ does involve certain time-independent integrals which must be evaluated; and their complexity increases with N.

Even though we have a set of integral power moments, we do not know that they represent a unique solution to the system in question. Hence, we are led to the "moment problem." For the evolution of an aerosol or droplet spectrum the statement of the moment problem given by Stieltjes (1894, 1895) is the one of importance. The statement is "find a bounded, non-decreasing function $F(x)$ in the interval $0\leqslant x<\infty$ such that its moments have a prescribed set of values

$$c_n=\int_0^\infty x^n dF(x), \quad n=0, 1, 2, \cdots." \qquad (7.2)$$

A detailed discussion of the moment problem of Stieltjes (SMP) is given in Shohat and Tamorkin (1943).

A sufficient condition that the SMP possesses a unique solution (Shohat and Tamarkin) is that c_n or $M_n(\tau)$ satisfy

$$\sum_{n=1}^{\infty} c_n^{-1/(2n)}=\infty. \qquad (7.3)$$

It can be shown (Drake, 1971) that (7.3) is satisfied for the kernel given in (3.4).

Eisen (1969) shows that if F is given by

$$dF=f(x)dx, \quad f(x)\geqslant 0, \qquad (7.4)$$

where

$$f(x)<M|x|^{a-1}\exp(-b|x|^c), \quad \text{for} \quad |x|\geqslant x_0, \qquad (7.5)$$

and where M, a and b are positive constants and x_0 is some predetermined quantity, then function F is unique if $c\geqslant\frac{1}{2}$ over the interval $(0,\infty)$. On the other hand, there are an infinite number of solutions if for sufficiently large x, we have

$$f(x)>\exp(-b|x|^c), \qquad (7.6)$$

where $0<c<\frac{1}{2}$. The importance of this result is that (7.5) gives an upper bound for all unique solutions of (2.6) which possess series expansions of the form given in (7.1). For example, for proper choices of M, a, b and c, the asymptotic form given in (4.7) is dominated by the upper bound given in (7.5).

The SMP can be interpreted in the light of the Mellin transform, (as shown by Sneddon, 1951). Suppose the sequence $\{c_n\}$ is replaced by a moment function $c(s)$, where $c(n)=c_{n-1}$ for $n=1, 2, \cdots$. The Mellin transform, $c(s)$, of the function $f(x)$ is defined by

$$c(s)=\int_0^\infty f(x)x^{s-1}dx. \qquad (7.7)$$

If $c(s)$ is known, then $f(x)$ can be obtained by applying the Mellin inversion formula. Stated as a theorem, this formula is:

"If the integral

$$\int_0^\infty x^{k-1}|f(x)|dx$$

is bounded for some $k>0$ and if $c(s)$ is defined by (7.7), then

$$f(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} c(s)x^{-s}ds, \tag{7.8}$$

where $a>k$."

The Mellin transform process can be used to obtain approximations for the solutions of (2.6). As an example, suppose that $f_0(x)$ and $\alpha(x,y)$ are given by

$$f_0(x)=e^{-x}, \quad \alpha(x,y)=1+\tfrac{1}{2}\sin(xy). \tag{7.9}$$

Thus, candidates for the upper and lower bounds of $dM_\nu(\tau)/d\tau$, $\nu\geqslant 1$, can be obtained by replacing $\alpha(x,y)$ in (3.1) by 3/2 and 1/2, respectively; that is

$$\frac{1}{4}\int_0^\infty\int_0^\infty [(x+y)^\nu-x^\nu-y^\nu]f(x,\tau)f(y,\tau)dxdy\leqslant\frac{dM_\nu(\tau)}{d\tau}$$

$$\leqslant\frac{3}{4}\int_0^\infty\int_0^\infty [(x+y)^\nu-x^\nu-y^\nu]f(x,\tau)f(y,\tau)dxdy. \tag{7.10}$$

The differential inequalities can be solved if ν is replaced by the positive integers n, and if the initial values of M_ν, given by

$$M_n(0)=\int_0^\infty x^n e^{-x}dx=n!, \tag{7.11}$$

are used. Hence, the corresponding upper and lower bounds for $M_n(\tau)$ are given by (see Drake, 1971)

$$n!\left(\frac{\tau+4}{4}\right)^{n-1}\leqslant M_n(\tau)\leqslant n!\left(\frac{3\tau+4}{4}\right)^{n-1}, \tag{7.12}$$

where $n=1, 2, 3, \cdots$.

If we now interpret n as a continuous variable, replace n by $s-1$, and replace $n!$ by $\Gamma(s)$, the bounds for $c(s)$ are given by

$$\left(\frac{\tau+4}{4}\right)^{s-2}\Gamma(s)\leqslant c(s)\leqslant\left(\frac{3\tau+4}{4}\right)^{s-2}\Gamma(s), \tag{7.13}$$

where $s\geqslant 2$. Because of the "-2" in the exponents of the upper and lower bounds, the inverse Mellin transform of (7.13) gives rational factors in the "upper" and "lower" bounds for $f(x,\tau)$ which are actually in the wrong direction. To correct this, we can rewrite (7.13) as

$$\left(\frac{2}{\tau+2}\right)^2\left(\frac{\tau+4}{4}\right)^s\Gamma(s)\leqslant\left(\frac{\tau+4}{4}\right)^{s-2}\Gamma(s)\leqslant c(s)$$

$$\leqslant\left(\frac{3\tau+4}{4}\right)^{s-2}\Gamma(s)\leqslant\left(\frac{2}{\tau+2}\right)^2\left(\frac{3\tau+4}{4}\right)^s\Gamma(s). \tag{7.14}$$

Taking the inverse Mellin transform of the extremes in (7.14) gives

$$L(x,\tau)=\left(\frac{2}{\tau+2}\right)^2\exp\left(-\frac{4x}{\tau+4}\right), \tag{7.15a}$$

$$U(x,\tau)=\left(\frac{2}{\tau+2}\right)^2\exp\left(-\frac{4x}{3\tau+4}\right), \tag{7.15b}$$

where L is the inverse of the extreme left-hand side of (7.14) and U the inverse of the extreme right side of (7.14). The ratio of U to L is given by

$$R(x,\tau)=\frac{U(x,\tau)}{L(x,\tau)}=\exp\left[\frac{8x\tau}{(\tau+4)(3\tau+4)}\right]. \tag{7.16}$$

Hence, $R(0,\tau)=R(x,\infty)=1$, $R(\infty,\tau)=\infty$ if $\tau<\infty$, and $R(x,\tau)\geqslant 1$ for all x and τ. Therefore, $U(x,\tau)\geqslant L(x,\tau)$ and U and L may be good candidates for the upper and lower bounds of $f(x,\tau)$, respectively. At least, U and L are reasonable approximations to $f(x,\tau)$, and the exponential terms in (7.24) give very good bounds for the asymptotic form of $f(x,\tau)$. The usefulness of the Mellin transform in obtaining solutions and bounds for solutions will be investigated in more detail in a later paper.

8. Conclusions

The major objective of this paper is to complement the known analytical results concerned with the pure coalescence equation. These new results, along with previously known results, will be of value in the parameterization of microphysical processes for large-scale, three-dimensional cloud models. The system which is considered here is given in nondimensional form in (2.6). In the current study special emphasis has been given to the analysis of the integral power moments of the size spectrum [see Eq. (2.7)], and the relationship between these moments and the collection kernel. The following list of conclusions were obtained from this study:

1) Using results from Thompson (1968), we derived a system of ordinary integro-differential equations for the moments $M_\nu(\tau)$, $\nu\geqslant 0$. In general, this system of equations is not a *closed* system and can not be solved inductively for the moments.

2) For special cases of the collection kernel and for integer power moments, this closure problem can be eliminated. Hence, if $\alpha(x,y)$ is given by (3.4), where A, B, C are non-negative constants and $\max[A,B,C]=1$, then all of the integer power moments, $M_N(\tau)$, $N=0, 1, 2, \cdots$, can be obtained by solving the above system inductively.

3) We found that whenever $C=0$ in (3.4), $M_0(\tau)>0$ and approached zero as $\tau\to+\infty$. However, when $C\neq 0$, $M_0(\tau)=0$ at $\tau=\tau_{\max}$; that is, the coalescence

process terminates in finite time. In fact, whenever C is near unity, τ_{max} is near 2; and when $C\to 0^+$, $\tau_{max}\to +\infty$.

4) We also found that whenever $C=0$, $M_2(\tau)\to +\infty$ as $\tau\to +\infty$. But if $C\neq 0$, $M_2(\tau)\to +\infty$ as $\tau\to\tau_\infty^-$. For C close to unity, τ_∞ is nearly 1 and as $C\to 0^+$, $\tau_\infty\to +\infty$.

5) Using realistic numerical values for cloud physics and conclusions 3) and 4), we found that $\alpha=xy$, or any kernel containing an xy term is probably a poor candidate for a realistic kernel.

6) The moment $M_0(\tau)$ is independent of the initial spectra only if α is given by (3.4); however, $M_0(\tau)$ is kernel-dependent. The moment $M_1(\tau)$ is equal to unity for all initial spectra and collection kernels, but the domain of definition of $M_1(\tau)$ is dependent upon the kernel, especially for kernels containing xy terms, or terms which dominate xy.

7) There is a strong indication that as the degree of homogeneity of the kernel increases, the dependence of the evolving spectra and their moments upon the initial spectra becomes more dominant.

8) Analyzing the exact solutions and the asymptotic forms derived by Scott (1968) for $\alpha=xy$, we find that their domain of validity is $0\leqslant\tau<\tau_\infty\leqslant 1$. This result is consistent with the theory of McLeod (1964) and the moment results obtained in this paper.

9) We considered several families of moment inequalities which are useful in deriving various properties of the solutions to (2.6). Some of the results derived from these inequalities are certain monotone properties of $M_\nu(\tau)$, various upper and lower bounds for $M_\nu(\tau)$, and certain relationships between the collection kernels and the moments. One kernel-dependent result which is of particular interest is that any kernel increasing as fast or faster than $\alpha=xy$ is probably a very poor candidate for known physical processes of coalescence.

10) Friedlander and Wang (1966) considered the existence of similarity solutions for (2.6) with homogeneous collection kernels. From the current study we conclude that if the kernel is homogeneous of degree $\geqslant 1$ there is probably no similarity solutions of the form specified by Friedlander and Wang.

11) From the known results for the Stieltjes moment problem we obtained an upper bound for the spectrum $f(x,\tau)$. This bound is consistent with the asymptotic forms of Scott (1968).

12) Finally, we introduced the idea that the moments $M_\nu(\tau)$ can be interpreted as the Mellin transform of $f(x,\tau)$. Two important possibilities arise from this interpretation. First, for certain kernels we can solve for the power moments and then take the inverse transform to get the corresponding spectrum. Second, we can find upper and lower bounds for the moments by solving differential inequalities and these in turn can be transformed to give reasonable estimates for the evolving spectrum $f(x,\tau)$. It appears that this technique may be very promising and will be investigated further.

APPENDIX

List of Symbols

a, A	constants
b, B	constants
c_n, C	constants
$c(s)$	moment function
D	constant
E	normalizing constant
$f(x,\tau)$	dimensionless size spectrum
$f_0(x)$	dimensionless initial size spectrum
$G(s)$	Laplace transform of $f(x)$
k	constant
$K(u,v)$	collection kernel
$L(x,\tau)$	an approximation of $f(x,\tau)$
$L[f(x)]$	Laplace transform
$L^{-1}[f(s)]$	inverse Laplace transform
$M_\nu(\tau)$	dimensionless integral power moments of $f(x,\tau)$
n, N	non-negative integers
$n(v,t)$	particle size density function, size spectrum
$\binom{N}{i}$	binomial coefficients
N_0	initial total number of particles
p	non-negative integer
$P(x,y)$	polynomial in x and y
q	non-negative integer
$Q(x,y)$	polynomial in x and y
$R(\tau), R(x,\tau)$	ratios of various functions
s	transformed variable corresponding to x
t	time
u, v	volume of particles
$U(x,\tau)$	an approximation of $f(x,\tau)$
v_0	initial mean volume of particles in a cloud
V	volume of a segment of a cloud
$w(x)$	weight function
x, y	dimensionless particle volumes
$\alpha(x,y)$	dimensionless collection kernel
$\Gamma(x)$	gamma function
$\delta(x)$	Dirac delta function
η	similarity variable
ν	non-negative constant
ξ	variable of integration
τ	dimensionless time
$\Phi(s)$	Laplace transform of ψ
$\psi(\eta)$	self-preserving spectrum
$\psi_i(x)$	orthogonal polynomials

REFERENCES

Beckenbach, E. F., and R. Bellman, 1965: *Inequalities*. New York, Springer-Verlag, 198 pp.

Berry, E. X. 1967: Cloud droplet growth by collection. *J. Atmos. Sci.*, **24**, 688–701.

Drake, R. L., 1971: The scalar transport equation of coalescence theory: Moments and kernels. NCAR Manuscript 252, Boulder, Colo.

Eisen, M., 1969: *Introduction to Mathematical Probability Theory.* Englewood Cliffs, N. J., Prentice-Hall, 535 pp.

Enukashvili, I. M., 1964a: On the solution of the kinetic coagulation equation. *Bull. Acad. Sci. USSR, Geophys. Ser.*, No. 10, 944–948.

——, 1964b: On the problem of a kinetic theory of gravitational coagulation in spatially heterogeneous clouds. *Bull. Acad. Sci. USSR, Geophys. Ser.*, No. 11, 1043–1045.

Feller, W., 1966: *An Introduction to Probability Theory and Its Applications,* Vol. 2. New York, Wiley, 626 pp.

Friedlander, S. K., 1961: Theoretical considerations for the particle size spectrum of the stratospheric aerosol. *J. Meteor.*, **18**, 753–759.

——, and C. S. Wang, 1966: The self-preserving particle size distribution for coagulation by Brownian motion. *J. Colloid Interface Sci.*, **22**, 126–132.

Golovin, A. M., 1963a: The solution of the coagulating equation for cloud droplets in a rising air current. *Bull. Acad. Sci. USSR, Geophys. Ser.*, No. 5, 482–487.

——, 1963b: On the spectrum of coagulating cloud droplets. II. *Bull. Acad. Sci. USSR, Geophys. Ser.*, No. 9, 880–884.

Levich, V. G., 1962: *Physicochemical Hydrodynamics.* Englewood Cliffs, N. J., Prentice-Hall, 700 pp.

McLeod, J. B., 1964: On the scalar transport equation. *Proc. London Math. Soc.*, **14**, 445–458.

Melzak, Z. A., 1957: A scalar transport equation. *Trans. Amer. Math. Soc.*, **85**, 547–560.

Müller, H., 1928: Zur allgemeinen Theorie der raschen Koagulation. *Kolloidchem. Bech.*, **27**, 223–250.

Saffman, P. G., and J. S. Turner, 1956: On the collision of drops in turbulent clouds. *J. Fluid Mech.*, **1**, 16–30.

Scott, W. T., 1968: Analytic studies of cloud droplet coalescence I. *J. Atmos. Sci.*, **25**, 54–65.

Shohat, J. A., and J. D. Tamorkin, 1943: *The Problem of Moments.* New York, Amer. Math. Soc., 138 pp.

Smoluchowski, M., 1917: Versuch einer mathematischen Theorie der koagulations-kinetic kolloider Losunger. *Z. Phys. Chem.*, **92**, 129–168.

Sneddon, I. N., 1951: *Fourier Transforms.* New York, McGraw-Hill, 542 pp.

Stieltjes, T. J., 1894: Recherches sur les fractions continues. *Ann. Fac. Sci. Toulouse,* **8**, T1–122.

——, 1895: Recherches sur les fractions continues. *Ann. Fac. Sci. Toulouse,* **9**, A5–47.

Swift, D. L., and S. K. Friedlander, 1964: The coagulation of hydrosols by Brownian motion and laminar shear flow. *J. Colloid Sci.*, **19**, 621–647.

Thompson, P. D., 1968: A transformation of the stochastic equation for droplet coalescence. *Proc. Intern. Conf. Cloud Phys.*, Toronto, 26–30 August, 115–126.

von Mises, R., 1964: *Mathematical Theory of Probability and Statistics.* New York, Academic Press, 694 pp.

Wang, Chiu-sen, and S. K. Friedlander, 1967: The self-preserving particle size distribution for coagulation by Brownian motion. II. Small particle slip connection and simultaneous shear flow. *J. Colloid Interface Sci.*, **24**, 1970–179.