6 Four-velocity and four-acceleration

We see immediately from the formula for addition of velocities (eq. 11) that ordinary velocity cannot be the spatial part of a four-vector. However, if we divide the four-vector Δx_μ with the scalar $\Delta\tau$, the result must be a four-vector. The four-vector

$$U_\mu = \frac{dx_\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx_\mu}{dt} = \gamma(u)(c, u_x, u_y, u_z) = \gamma(u)(c, \mathbf{u}), \tag{26}$$

we call the *four-velocity.* Here $\mathbf{u}$ is the velocity of our particle relative to our inertial reference system. We see that for small velocities, the spatial part of the four-velocity is the ordinary velocity while the temporal component is the speed of light. In an inertial system which is instantaneously at rest with respect to the particle, we see that the four-velocity is $(c, 0, 0, 0)$. Hence the scalar $-U_0^2 + U_i U_i = -c^2$ always for all particles in all inertial reference frames.

Now, let us introduce a new four-vector, the four-acceleration, the four-velocity differentiated with respect to the proper time (since we again in principle divide a four vector by a scalar, the result must be a four-vector),

$$A_\mu = \frac{dU_\mu}{d\tau} = \frac{dt}{d\tau}\frac{dU_\mu}{dt} = \gamma(u)\frac{d}{dt}(c\gamma(u), \gamma(u)\mathbf{u}). \tag{27}$$

If we introduce the ordinary acceleration, $\mathbf{a} = d\mathbf{u}/dt$, we see that

$$\frac{d\gamma}{dt} = \frac{1}{2c^2}\gamma^3\frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) = \frac{\gamma^3}{c^2}\mathbf{u}\cdot\mathbf{a}. \tag{28}$$

We then have that

$$A_\mu = \gamma(u)\left(\frac{\gamma^3}{c}\mathbf{u}\cdot\mathbf{a}, \gamma\mathbf{a} + \frac{\gamma^3}{c^2}(\mathbf{u}\cdot\mathbf{a})\mathbf{u}\right). \tag{29}$$

Now, using the well known vector relation

$$(\mathbf{u}\cdot\mathbf{a})\mathbf{u} = \mathbf{u}\times(\mathbf{u}\times\mathbf{a}) + (\mathbf{u}\cdot\mathbf{u})\mathbf{a}, \tag{30}$$

we find that the four-acceleration can be written as

$$A_\mu = \gamma^4\left(\vec{\beta}\cdot\mathbf{a}, \mathbf{a} + \vec{\beta}\times(\vec{\beta}\times\mathbf{a})\right). \tag{31}$$

By doing some algebra, we find that the scalar $A^\mu A_\mu$ is given by

$$A^\mu A_\mu = -A_0A_0 + A_iA_i = \gamma^6\left(a^2 - (\vec{\beta}\times\mathbf{a})^2\right), \tag{32}$$

which therefore is the same in all inertial reference systems. In a system instantaneously at rest with respect to the particle, $\mathbf{u} = 0$ and $\mathbf{a} = \mathbf{a}_0$. In this frame, the four-acceleration is $(0, \mathbf{a}_0)$, and hence we see that

$$A^\mu A_\mu = -A_0A_0 + A_iA_i = a_0^2. \tag{33}$$

We see immediately that the Lorentz invariant scalar product of the four-velocity and the four-acceleration is zero in the frame instantaneously at rest with respect to the particle, and hence it is always zero, i.e.,

$$A^\mu U_\mu = -A_0 U_0 + A_i U_i = 0. \tag{34}$$

Let us now look at a particle, e.g., a rocket, starting at rest in the origin of one inertial reference system and being constantly accelerated along the x-axis at every instant with a constant acceleration g with respect to an inertial reference frame instantly at rest with respect to the particle. In our original reference frame, we then have that $A^\mu A_\mu = \gamma^6 a^2 = a_0^2 = g^2$, i.e.,

$$\gamma^3 \frac{du}{dt} = g. \tag{35}$$

But from equation (28) we see that

$$\frac{d}{dt}(\gamma u) = \gamma \frac{du}{dt} + u\frac{d\gamma}{dt} = \gamma^3 \frac{du}{dt}\left(\gamma^{-2} + \frac{u^2}{c^2}\right) = \gamma^3 \frac{du}{dt}. \tag{36}$$

Therefore

$$\frac{d}{dt}\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = g, \tag{37}$$

which by integration gives

$$u = \frac{dx}{dt} = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}} \tag{38}$$

and

$$x = \frac{c^2}{g}\int_0^{gt/c} \frac{xdx}{\sqrt{1+x^2}} = \frac{c^2}{g}\sqrt{1+\frac{g^2t^2}{c^2}} - \frac{c^2}{g}. \tag{39}$$

Then we have that

$$\left(x + \frac{c^2}{g}\right)^2 - c^2t^2 = \frac{c^4}{g^2}, \tag{40}$$

which is the equation for a hyperbola in the ct-x plane. In special relativity such motion with constant acceleration (defined relative to the instantaneous inertial reference frame) is called *hyperbolic motion.* We see that for