Principles of special relativity

- **Introduction.**
  Introducing optical devices, we saw that the electron-photon perturbation Hamiltonian is given by (see Lecture Notes, Part 3, page 267, Eq. (252)):

$$H_{phot} = \frac{ie\hbar}{mc}\mathbf{A}\cdot\nabla \ . \tag{1}$$

  This interaction term can be obtained by starting from the free-electron Hamiltonian $\mathbf{p}^2/(2m)$ and replacing the electron momentum $\mathbf{p}$ with $\mathbf{p} - e\mathbf{A}/c$. (The factor of c in the denominators appears when using Gaussian units, which are more convenient in this context and we shall use them here). The reason behind this substitution relies on some basic principles of special relativity. Let's see how.

- **Galilean invariance and Maxwell's equations.**
  A soon as Maxwell's equation were formulated, it became clear that there was a major difference with respect to Newton's law. Let's start with Newton's second law,

$$\mathbf{F} = m\mathbf{a} \qquad \text{or} \qquad \mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2} \ . \tag{2}$$

  and consider this same equation as could be written by somebody (called an 'observer') which is moving with respect to us with uniform velocity $\mathbf{u}$. Let's assume that we and the other observer use a reference frame with parallel x, y, and z axes, that the other observer moves along the x axis, so that $\mathbf{u} = (u,0,0)$, and that the origin of the two reference frames coincide at a fixed instant in time which we take $t = 0$. Then, calling (x,y,z) our frame and (x',y',z') the other observer's frame, an object located at a point (x,y,z) in our frame will be located at a point (x',y',z') in the observer's frame, such that:

$$\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \end{aligned} \ . \tag{3}$$

Therefore, expressing Newton's second law in the 'primed' frame,

$$\mathbf{F} \;=\; m\frac{d^2\mathbf{x}'}{dt^2} \;=\; m\frac{d^2(\mathbf{x}-\mathbf{u}t)}{dt^2} \;=\; m\frac{d^2\mathbf{x}}{dt^2} \;-\; \frac{d\mathbf{u}}{dt} \;=\; m\frac{d^2\mathbf{x}}{dt^2} \;. \tag{4}$$

In other words, Newton's law retains the same algebraic form in all frames which are moving with uniform velocity with respect to our frame. This principle can be generalized by saying that the laws of mechanics are valid in all 'inertial frames'. An observer, by performing experiments, cannot tell whether he/she is moving with respect to other inertial frames. This is the principle of Galilean relativity, since it was proposed (noted, discovered, invented?... the choice of a term is a a matter of deep philosophical discussions) by Galileo.
Consider now Maxwell's equations. For simplicity, let's just consider a wave equation

$$\left(\nabla^2 \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\;\psi \;=\; 0 \;. \tag{5}$$

Let's apply the transformation Eq. (3). Since in the 'unprimed' frame $\psi(x,y,z,t) = \psi(x'+ut,y,z,t)$, so that $\partial\psi/\partial x' = \partial\psi/\partial x + u\partial\psi/\partial t$, we find:

$$\left(\nabla^2 \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \frac{2u}{c^2}\frac{\partial^2}{\partial x\partial t} \;-\; \frac{u^2}{c^2}\frac{\partial^2}{\partial x^2}\right)\;\psi \;=\; 0 \;. \tag{6}$$

What a mess! The form of the equation has been completely altered by the transformation! In hindsight, we already knew it had to be so: After all, magnetic fields caused by moving charges must disappear when we use a frame in which the charges are at rest. Therefore, the $\mathbf{E}$ and $\mathbf{B}$ fields do not transform correctly under the Galilean transformation Eq. (1). Moreover, the Lorentz force depends explicitly on the velocity of the particle, so that the form of the equation will differ in a different inertial frame. Historically, this is also related to the difficulty of understanding electromagnetic waves: Sound waves are oscillations of the medium in which they propagate. But electromagnetic waves are oscillations of what?

In order to fix the situation we have three alternatives we can choose from:

1. Maxwell's equations are wrong. The correct equations, yet to be discovered, are invariant under Galilean transformations.
2. Galilean invariance is valid for mechanics, not for electromagnetism. This is the historical solution before Einstein: The 'ether' determines the existence of the 'absolute frame' in which the ether is at rest and Mawxwell's equations hold.
3. Galilean invariance is wrong. There is a more general invariance – yet to be dsicovered – which preserves the form of Maxwell's equation. Classical mechanics is incorrect and must be reformulated so that it is invariant under this new transformation.

Having to chose between thrashing Maxwell (option 1) or Newton (option 3), physicists chose the easier option 2. Einstein, instead, decided to follow the third option, guided by two postulates:

1. **Postulate of relativity:** All physical laws must 'look the same' in all frames moving with uniform velocity with repect to each other.
2. **Postulate of the constancy of the speed of light:** The speed of light is the same (numerically the same!) independent of the velocity of the observer or of its source. This stems logically from the Michelson-Morley experiment of 1887, but the result could have been explained 'saving' ether and using the Lorentz-FitzGerald contractions.

Armed with these postulates, Einstein set to build a new set of transformations between inertial frames. Maxwell's equation are now invariant under this new set of transformations, but Newtonian mechanics has to be modified: If two frames move at a relative speed much smaller than the speed of light, the 'new' transformations approach the usual Galilean transfomation and Newton is approximately correct. But for relative velocities approaching the speed of light, the laws of mechanics deviate enormously from Newton's laws.

It is impossible to pay justice to special relativity in such a short time. We shall only discuss those few concepts which are required to answer our original question of why we perform the substitution $\mathbf{p} \to \mathbf{p} - (e/c)\mathbf{A}$.

- **Lorentz transformations.**

Consider the same two frames ('primed' K' and 'unprimed' K frames) considered above. Assume now that at $t = 0$ (the time at which the origins of the two frames coincided... at least when looking at our clock...!) a ray of light was emitted from the origin. Since light travels at the same speed in both frames, we must have for the