Solar Energy Vol. 31, No. 5, pp. 439–444, 1983
Printed in Great Britain.

0038–092X/83 $3.00+.00
Pergamon Press Ltd.

SOLAR GEOMETRY FOR FIXED AND TRACKING SURFACES

J. E. BRAUN and J. C. MITCHELL†
Solar Energy Laboratory, University of Wisconsin, Madison, WI 53706, U.S.A.

(*Received* 26 *April* 1982; *accepted* 4 *January* 1983)

Abstract—A general expression for the solar radiation incidence angle in terms of the slope and surface azimuth is derived for both fixed and tracking surfaces. Using this expression, relationships for the slope and azimuth of optimally tracked one- and two-axis tracking surfaces are developed. This information is necessary in determining incident solar radiation based upon horizontal measurements.

1. INTRODUCTION

Radiation measurements are generally available for hourly intervals on horizontal surfaces. In order to determine the hourly incident radiation on a surface of any orientation, it is necessary to evaluate the ratio of incident radiation on the tilted surface to that on a horizontal surface considering beam, sky diffuse, and ground reflected radiation separately[1, 2]. Evaluating beam radiation on a tilted surface requires knowledge of the position of the sun relative to both the surface normal and vertical. The incident sky diffuse and ground reflected radiation depend upon the view factors between the surface and the sky and the surface and the ground. Both factors are functions of the position of the surface relative to the horizontal. The geometry necessary to analyze the problem is described in terms of three angles as shown in Fig. 1. The slope of the surface, β, is the angle between the surface normal and the vertical. The incidence angle, θ, is measured between a ray from the sun and the surface normal. The zenith angle, θ_z, is the angle between the vertical and a ray from the sun (i.e. the incidence angle for a horizontal surface).

Relationships for the incidence angle have been derived in terms of the position of the surface and the sun[2, 3]. They are applicable whether the surface is fixed or tracking so long as the position of the surface is known at any instant. Expressions for incidence angle are also available for optimally tracked surfaces that rotate about a single axis that is oriented either north–south or east–west or that have two axes of rotation[2, 4, 5]. These relationships neither require nor yield information about the position of the surface as a function of time. As a result, it is not possible to evaluate the incident sky diffuse and ground reflected radiation. The energy output of a concentrating collector is not sensitive to diffuse radiation, therefore the position of the surface is not an essential piece of information for modeling this collector type. The incident diffuse radiation would be of interest, however, in evaluating the overall collector efficiency for performance comparisons between different collector designs. In this paper, an expression for incidence angle in terms of the position of the surface is derived which is general for both fixed and tracking surfaces. Using this expression, relationships for the position of optimally tracked single-axis trackers of any axis orientation and two axis trackers are developed.

The analysis in this paper utilizes trigonometric relations for spherical triangles. A good summary of spherical trigonometry is given in Ref. [6]. For those not interested in the derivation, the resulting expressions are given in Tables 1 and 2. In addition, a sample calculation is presented in Section 4.

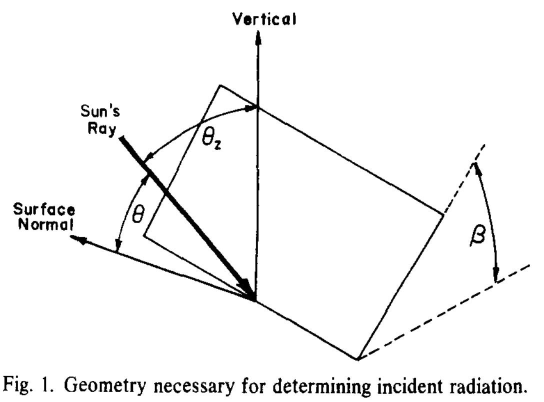

Fig. 1. Geometry necessary for determining incident radiation.

†Laboratory for Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

2. ANGLE OF INCIDENCE

Figure 2 is useful in deriving an expression for the solar incidence angle for an arbitrarily oriented surface. Points v, n, and s are all unit distances from the origin o. Vector $\vec{ov}$ is a vertical, $\vec{on}$ is normal to the surface, and $\vec{os}$ points in the direction of the line of sight to the sun. The arcs shown connecting the three points lie on the sphere of unit radius with origin o. Hence, arc length $\widehat{nv}$ is the surface slope, β, $\widehat{sn}$ is the incidence angle, θ, and $\widehat{sv}$ is the zenith angle, θ_z. The solar azimuth, γ_s, is the angle between the local meridian and the projection of the line of sight of the sun into the horizontal plane. The surface azimuth, γ, is the angle between local meridian and the horizontal projection of the surface normal. The sign convention for both solar and surface azimuths is that zero aximuth denotes facing the equator, west is

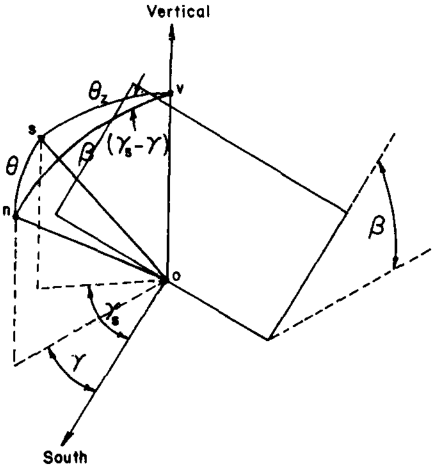

Fig. 2. Surface–Sun geometry.

positive, and east is negative. Applying the Law of Cosines to spherical triangle *nvs* gives an expression for the cosine of the incidence angle.

$$\cos\theta = \cos\theta_z \cos\beta + \sin\theta_z \sin\beta \cos(\gamma_s - \gamma) \quad (1)$$

Expressions for solar zenith and azimuth can be derived through the use of a unit sphere depicting the earth as shown in Fig. 3. The center of the earth is point O, N is the North Pole, P is the point of observation at hour angle ω, and Q is the point where the sun is directly overhead. The hour angle is measured from solar noon, with mornings negative and afternoons positive (ω is negative in Fig. 3). Vector $\vec{OP}$ is the vertical and $\vec{OQ}$ points directly at the sun, hence $< POQ$ and $\widehat{PQ}$ are equal to the solar zenith. The declination, δ, referred to in Fig. 3 can be approximated from the equation of Cooper[7] using the day of the year, n.

$$\delta = 23.45 \sin(360(284 + n)/365). \quad (2)$$

By the Law of Cosines for spherical triangle NPQ, the cosine of the zenith angle can be expressed as

$$\cos\theta_z = \sin\delta \sin\phi + \cos\delta \cos\phi \cos\omega. \quad (3)$$

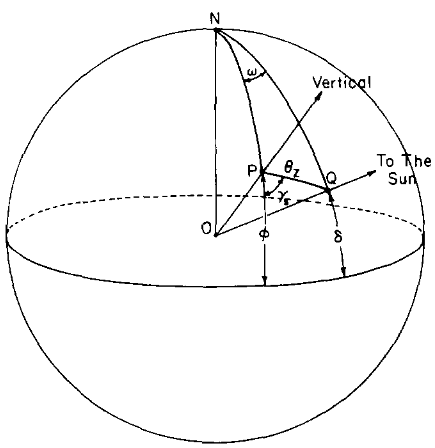

Fig. 3. Earth–Sun geometry.

The Law of Sines for NPQ yields

$$\sin\gamma_s = \frac{\sin\omega \cos\delta}{\sin\theta_z}. \quad (4)$$

The solar azimuth is generally negative in the morning, zero at noon, and positive in the afternoon. The exception occurs in the tropics when the declination exceeds the latitude in the northern hemisphere or is less than the latitude in the southern hemisphere, i.e. $\phi(\phi - \delta) < 0$. In this case, the sun is in the direction opposite the equator (at an azimuth angle of 180°) at noon.

One additional angle is required to determine the quadrant of γ_s. This is the absolute value of the hour angle when the sun is due east (or, equivalently, west). Applying spherical trigonometry to NPQ for $\gamma_s = 90°$ will give

$$\sin\delta = \sin\phi \cos\theta_z \quad (5)$$

which can be simplified using eqn (3) to obtain

$$\omega_{ew} = \cos^{-1}(\cot\phi \tan\delta). \quad (6)$$

The magnitude of γ_s is less than 90° whenever either $\phi(\phi - \delta) \geq 0$ and $|\omega| < \omega_{ew}$ or $\phi(\phi - \delta) < 0$ and $|\omega| > \omega_{ew}$. Thus,

$$\gamma_s = \sigma_{ew} \cdot \sigma_{ns} \cdot \gamma_{so} + \left(\frac{1 - \sigma_{ew} \cdot \sigma_{ns}}{2}\right) \cdot \sigma_\omega \cdot 180° \quad (7)$$

where

$$\gamma_{so} = \sin^{-1}\left(\frac{\sin\omega \cos\delta}{\sin\theta_z}\right)$$

$$\sigma_{ew} = \begin{cases} 1 \text{ if } |\omega| \leq \omega_{ew} \\ -1 \text{ otherwise} \end{cases}$$

$$\sigma_{ns} = \begin{cases} 1 \text{ if } \phi(\phi - \delta) \geq 0 \\ -1 \text{ otherwise} \end{cases}$$

$$\sigma_w = \begin{cases} 1 \text{ if } \omega \geq 0 \\ -1 \text{ otherwise} \end{cases}.$$

Table 1 summarizes the expressions necessary for evaluating the solar incidence angle derived in this section.

3. TRACKING SURFACES

Equation (1) gives the angle of incidence for both fixed and tracking surfaces in terms of surface slope and azimuth. For tracking surfaces, it is necessary to have a functional relationship for β and γ. In the analysis that follows, it is assumed that the slope and/or azimuth are continually adjusted so that incident beam radiation is always maximized. This is equivalent to maximizing $\cos\theta$.

Figure 4 illustrates a single-axis tracking surface with a fixed slope and variable azimuth rotating about a vertical axis. In this case, beam radiation is maximized when

Table 1. Position of the sun

Description	Mathmatical Expression
Incidence Angle	$\theta = \cos^{-1}[\cos\theta_z \cos\beta + \sin\theta_z \sin\beta \cos(\gamma_s - \gamma)]$
Zenith Angle	$\theta_z = \cos^{-1}[\sin\delta \sin\phi + \cos\delta \cos\phi \cos\omega]$
Solar Azimuth Angle	$\gamma_s = \sigma_{ew}\cdot\sigma_{ns}\cdot\gamma_{so} + \left(\frac{1 - \sigma_{ew}\cdot\sigma_{ns}}{2}\right)\cdot\sigma_\omega\cdot 180°$
	where,
	$\gamma_{so} = \sin^{-1}\left[\frac{\sin\omega \cos\delta}{\sin\theta_z}\right]$
	$\sigma_{ew} = \begin{cases} 1 \text{ if } \lvert\omega\rvert < \omega_{ew} \\ -1 \text{ otherwise} \end{cases}$
	$\sigma_{ns} = \begin{cases} 1 \text{ if } \phi(\phi - \delta) \geq 0 \\ -1 \text{ otherwise} \end{cases}$
	$\sigma_\omega = \begin{cases} 1 \text{ if } \omega \geq 0 \\ -1 \text{ otherwise} \end{cases}$
	$\omega_{ew} = \cos^{-1}(\cot\phi \tan\delta)$
Declination	$\delta = 23.45 \sin[360(284 + n)/365]$

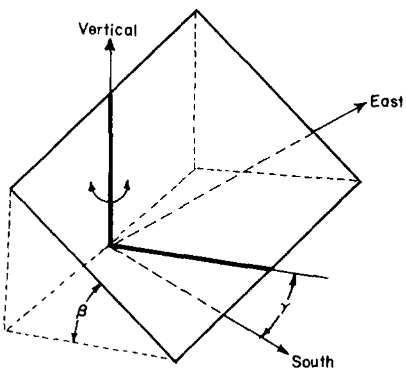

Fig. 4. Vertical axis tracking surface.

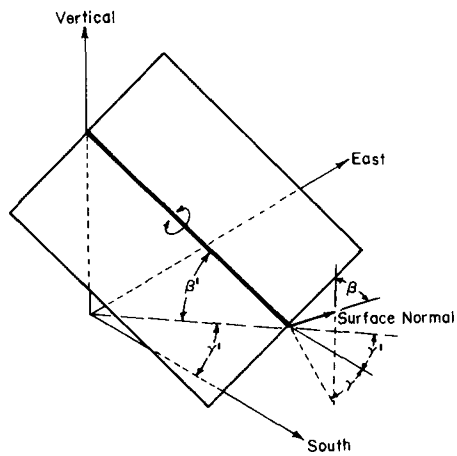

Fig. 5. Single-axis tracking; surface parallel to axis.

$$\frac{d(\cos\theta)}{d\gamma} = -\sin\theta_z \sin\beta \sin(\gamma_s - \gamma) = 0. \tag{8}$$

This is satisfied when the surface azimuth equals the solar azimuth.

The case of a surface rotating about a single axis that is always parallel to the surface is shown in Fig. 5. If the axis is horizontal, then beam radiation is maximized when

$$\frac{d(\cos\theta)}{d\beta} = \sin\theta_z \cos(\gamma_s - \gamma)\cos\beta - \cos\theta_z \sin\beta = 0 \tag{9}$$

or when

$$\tan\beta = \tan\theta_z \cos(\gamma - \gamma_s). \tag{10}$$

The surface slope is defined as being between 0 and 180° so that a general expression for β is given as

$$\beta = \beta_0 + \sigma_\beta \cdot 180° \tag{11}$$

where

$$\beta_0 = \tan^{-1}(\tan\theta_z \cos(\gamma - \gamma_s))$$
$$\sigma_\beta = \begin{cases} 0 \text{ if } \beta_0 \geq 0 \\ 1 \text{ otherwise} \end{cases}. \tag{12}$$

The surface azimuth is given in terms of the axis azimuth as

$$\gamma = \gamma' + 90° \text{ if } \gamma_s - \gamma' \geq 0° \quad (12)$$

$$\gamma = \gamma' - 90° \text{ if } \gamma_s - \gamma' < 0°. \tag{13}$$

If a surface is tracked about a single axis that is always parallel to the surface, but is not vertical or horizontal, both the azimuth and slope of the surface vary with time. Beam radiation is always maximized when the sun lies in the plane that is perpendicular to the surface and contains the tracking axis. The three spherical triangles shown in Fig. 6 are used to analyze this problem. Vectors $\vec{ov}$, $\vec{on}$ and $\vec{os}$ are as previously defined, vector $\vec{on}'$ and angle θ' are the normal and incidence angle, respectively, for a surface with slope and azimuth equal to those of the axis (i.e. β' and γ').

Applying the Law of Cosines to each of these spherical triangles results in the following three relations:

$$\cos\epsilon = \frac{\cos\beta}{\cos\beta'} \tag{14}$$

$$\cos\theta = \frac{\cos\epsilon}{\cos\theta'} \tag{15}$$

$$\cos\epsilon = \cos\beta'\cos\beta + \sin\beta'\sin\beta\cos(\gamma-\gamma'). \tag{16}$$

By inserting eqn (14) into eqn (16), a relationship between the surface slope and azimuth can be found as

$$\tan\beta = \frac{\tan\beta'}{\cos(\gamma-\gamma')} \tag{17}$$

Substituting eqn (14) into eqn (15), solving for $\cos\theta$ and equating the result with eqn (1) yields

$$\frac{\cos\theta\cos\beta'}{\cos\beta} = \cos\beta\cos\theta_z + \sin\beta\sin\theta_z\cos(\gamma_s-\gamma). \tag{18}$$

The trigonometric identity, $1/\cos^2\beta = 1 - \tan^2\beta$ can be inserted into eqn (18) and the result rearranged to give

$$\cos\theta_z - \cos\theta'\cos\beta' = \cos\theta'\cos\beta'\tan^2\beta - \sin\theta_z\cos(\gamma-\gamma_s)\tan\beta. \tag{19}$$

If eqn (17) is substituted into the preceding expression, then the only unknown remaining is the surface azimuth, γ. Algebraic manipulation of this result can give

$$\tan(\gamma-\gamma') = \frac{\sin\theta_z\sin(\gamma_s-\gamma')}{\cos\theta'\sin\beta'}. \tag{20}$$

Recognizing the fact that the sign of $\gamma-\gamma'$ is always the same as the sign of $\gamma_s-\gamma'$, a general expression for the surface azimuth can be written as

$$\gamma = \gamma_0 + \sigma_{\gamma 1}\cdot\sigma_{\gamma 2}\cdot 180° \tag{21}$$

where

$$\gamma_0 = \gamma' + \tan^{-1}\left[\frac{\sin\theta_z\sin(\gamma_s-\gamma')}{\cos\theta'\sin\beta'}\right]$$

$$\sigma_{\gamma 1} = \begin{cases} 0 \text{ if } (\gamma_0-\gamma')(\gamma_s-\gamma') \ge 0 \\ 1 \text{ otherwise} \end{cases}$$

$$\sigma_{\gamma 2} = \begin{cases} 1 \text{ if } (\gamma_s-\gamma') \ge 0 \\ -1 \text{ otherwise} \end{cases}.$$

Since the surface slope is always between 0 and 180°, a general expression for β can be obtained from eqn (17) as

$$\beta = \beta_0' + \sigma_\beta\cdot 180° \tag{22}$$

where

$$\beta_0' = \tan^{-1}\left[\frac{\tan\beta'}{\cos(\gamma-\gamma')}\right]$$

$$\sigma_{\beta'} = \begin{cases} 0 \text{ if } \beta_0' \ge 0 \\ 1 \text{ otherwise} \end{cases}.$$

For a two-axis tracking surface, the surface will be adjusted such that the sun is always at normal incidence to maximize beam radiation, i.e.

$$\frac{\partial(\cos\theta)}{\partial\beta} = 0 \tag{23}$$

$$\frac{\partial(\cos\theta)}{\partial\gamma} = 0. \tag{24}$$

These equations are satisfied when $\gamma = \gamma_s$ and $\beta = \theta_z$.

Table 2 summarizes the relationships for the positions of the optimally tracked surfaces considered here.

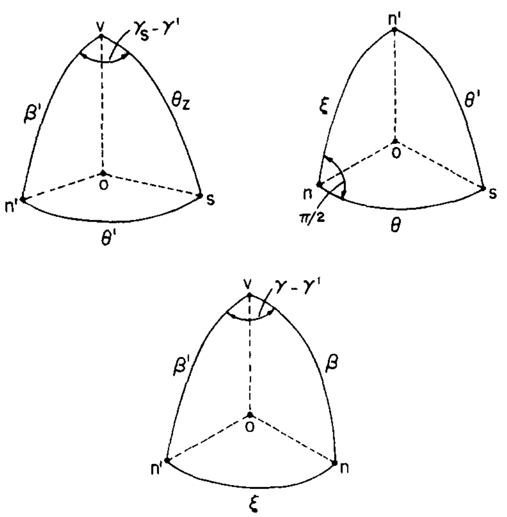

Fig. 6. Spherical triangles for analyzing single-axis trackers.

4. EXAMPLE CALCULATION

As an illustration of the use of the relationships derived in this paper, the solar incidence angle will be determined for both a fixed and a single-axis tracking surface of the type shown in Fig. 5. The azimuth and slope of the fixed surface and the tracking axis are 15 and 45°, respectively. The calculations are to be done for 4 July at 2 hr before solar noon at a 45°north latitude.

Sun's position

4 July is the 185th day of the year. Evaluating the declination of the sun using eqn (2) results in

$$\delta = 23.45\sin\frac{360(284+185)}{365} = 22.89°.$$

Table 2. Positions of optimally tracked surfaces

Tracking Type	Surface Azimuth	Surface Slope
Vertical axis, fixed slope	$\gamma = \gamma_s$	---
Horizontal axis, surface parallel to axis	$\gamma = \gamma' + 90°$ if $\gamma_s - \gamma' \geq 0$ $\gamma = \gamma' - 90°$ if $\gamma_s - \gamma' < 0$	$\beta = \beta_0 + \sigma_\beta \cdot 180°$ where, $\beta_0 = \tan^{-1}(\tan\theta_z \cos(\gamma - \gamma_s))$ $\sigma_\beta = \begin{cases} 0 \text{ if } \beta_0 \geq 0 \\ 1 \text{ otherwise} \end{cases}$
Sloped axis, surface parallel to axis	$\gamma = \gamma_0 + \sigma_{\gamma 1} \cdot \sigma_{\gamma 2} \cdot 180°$ where, $\gamma_0 = \gamma' + \tan^{-1}\left[\frac{\sin\theta_z \sin(\gamma_s - \gamma')}{\cos\theta' \sin\beta'}\right]$ $\sigma_{\gamma 1} = \begin{cases} 0 \text{ if } (\gamma_0 - \gamma')(\gamma_s - \gamma') \geq 0 \\ 1 \text{ if otherwise} \end{cases}$ $\sigma_{\gamma 2} = \begin{cases} 1 \text{ if } (\gamma_s - \gamma') \geq 0 \\ -1 \text{ otherwise} \end{cases}$	$\beta = \beta_0' + \sigma_\beta' \cdot 180°$ where, $\beta_0' = \tan^{-1}\left[\frac{\tan\beta'}{\cos(\gamma - \gamma')}\right]$ $\sigma_{\beta'} = \begin{cases} 0 \text{ if } \beta_0' \geq 0 \\ 1 \text{ otherwise} \end{cases}$
Two-axis tracking	$\gamma = \gamma_s$	$\beta = \theta_z$

The sun travels 15° per hr so that the hour angle for this example is

$$\omega = -30°.$$

The zenith angle is evaluated using eqn (3) as

$$\theta_z = \cos^{-1}(\sin(22.89°)\sin(45°) + \cos(22.89°)\cos(45°)\cos(-30°)) + 32.95°.$$

The solar azimuth angle is found using eqn (7). Since $\phi(\phi - \delta) > 0$ and $\omega < 0$, $\sigma_{ns} = 1$ and $\sigma_\omega = -1$. In order to determine σ_{ew}, it is necessary to evaluate the hour angle when the sun is due west with eqn (6) as

$$\omega_{ew} = \cos^{-1}(\cot(45°)\tan(22.89°)) = 65.03°.$$

Since $|\omega| < \omega_{ew}$ and $\sigma_{ew} < 1$, the solar azimuth is

$$\gamma_s = \sin^{-1}\left(\frac{\sin(-30°)\cos(22.89°)}{\sin(32.95°)}\right) = -57.87°.$$

Fixed surface incidence angle

The solar incidence angle is evaluated with eqn (1) as

$$\theta = \cos^{-1}(\cos(30.95°)\cos(45°) + \sin(30.95° \sin(45°)\cos(-57.87 - 15°)) = 44.48°.$$

Tracking surface incidence angle

The surface azimuth and slope of the tracking surface are found using eqns (21) and (22), respectively. Since $(\gamma_0 - \gamma')(\gamma_s - \gamma') > 0$ and $\beta_o' > 0$, then

$$\gamma = 15° + \tan^{-1}\left(\frac{\sin(30.95°)\sin(-57.87° - 15°)}{\cos(44.48°)\sin(45°)}\right) = -29.25°$$

$$\beta = \tan^{-1}\left(\frac{\tan(45°)}{\cos(-29.25 - 15°)}\right) = 54.39°.$$

The incidence angle for the tracking surface is

$$\theta = \cos^{-1}(\cos(30.95°)\cos(54.39°) + \sin(30.95°)\sin(45°)\cos(-57.87 + 29.25)) = 35.06°.$$

NOMENCLATURE

- n day of year; 1 corresponds to 1 January, 365 to 31 December
- β surface slope; angle between the surface normal and the vertical
- β' slope of the tracking axis; angle between the axis line and the projection of the axis line into the horizontal plane
- γ surface azimuth; angle between the projection of the normal to the surface into the horizontal plane and the local meridian. Zero azimuth is facing the equator, west positive, east negative
- γ' azimuth of the tracking axis; angle between the project of the axis line onto the horizontal plane and local meridian. Same sign convention as surface azimuth

γ_s solar azimuth; angle between the line of sight of the sun into the horizontal plane and the local meridian. Same sign convention as surface azimuth
δ declination; angular position of the sun at solar noon with respect to the plane of the equator, north positive, south negative
θ angle of incidence; angle between a direct ray from the sun and the surface normal
θ' angle of incidence for a surface with a slope and azimuth equal to those of the tracking axis
θ_z zenith angle; angle between a direct ray from the sun and the vertical
ϕ latitude; angular location north or south of the equator, north positive, south negative
ω hour angle; angular displacement of the sun east or west of the local meridian due to rotation of the earth on its axis, morning negative, afternoon positive
ω_{ew} the absolute value of the hour angle when the sun is directly east or west
$\overrightarrow{on}$ unit vector surface normal
$\overrightarrow{on}'$ unit vector normal to a surface with slope and azimuth equal to those of the tracking axis
$\overrightarrow{os}$ unit vector originating on the surface that lies along the line of sight to the sun
$\overrightarrow{ov}$ unit vertical vector originating on the surface
$\overrightarrow{OP}$ unit vertical vector originating at the center of the earth
$\overrightarrow{OQ}$ unit vector originating at the center of the earth that lies along the line of the sight to the sun

REFERENCES

1. B. Y. H. Liu and R. C. Jordan, The long-term average performance of flat-plate solar energy collectors. *Solar Energy* **7**, 53 (1963).
2. J. A. Duffie, and W. A. Beckman, *Solar Engineering of Thermal Processes.* Wiley-Interscience, New York (1980).
3. F. Benford, and J. E. Bock, A time analysis of sunshine. *Trans. Am. Illumination Engng Soc.* **34**, 200 (1939).
4. J. A. Eibling, R. E. Thomas and B. A. Landry, An investigation of multiple-effect evaporation of saline waters by steam from solar radiation. Report to the Office of Saline Water, U.S. Department of Interior (1953).
5. M. Collares-Pereira and A. Rabl, Derivation of method for predicting long term average energy delivery of solar collectors. *Solar Energy* **23**, 223 (1979).
6. *CRC Standard Mathematic Tables.* CRC Press, Cleveland, Ohio (1974).
7. P. I. Cooper, The absorption of solar radiation in solar stills. *Solar Energy* **12**, 3 (1969).