We can expand $\psi(\mathbf{x})$ in terms of $a^s_\mathbf{p}$ and $b^s_\mathbf{p}$ as before (Eq. (3.87)). The creation and annihilation operators must now obey

$$\left\{a^r_\mathbf{p}, a^{s\dagger}_\mathbf{q}\right\} = \left\{b^r_\mathbf{p}, b^{s\dagger}_\mathbf{q}\right\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs} \tag{3.97}$$

(with all other anticommutators equal to zero) in order that (3.96) be satisfied. Another computation gives the Hamiltonian,

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \Big(E_\mathbf{p} a^{s\dagger}_\mathbf{p} a^s_\mathbf{p} - E_\mathbf{p} b^{s\dagger}_\mathbf{p} b^s_\mathbf{p}\Big),$$

which is the same as before; $b^{s\dagger}_\mathbf{p}$ still creates negative energy. However, the relation $\{b^r_\mathbf{p}, b^{s\dagger}_\mathbf{q}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$ is symmetric between $b^r_\mathbf{p}$ and $b^{s\dagger}_\mathbf{q}$. So let us simply redefine

$$\tilde{b}^s_\mathbf{p} \equiv b^{s\dagger}_\mathbf{p}; \qquad \tilde{b}^{s\dagger}_\mathbf{p} \equiv b^s_\mathbf{p}. \tag{3.98}$$

These of course obey exactly the same anticommutation relations, but now the second term in the Hamiltonian is

$$-E_\mathbf{p} b^{s\dagger}_\mathbf{p} b^s_\mathbf{p} = +E_\mathbf{p} \tilde{b}^{s\dagger}_\mathbf{p} \tilde{b}^s_\mathbf{p} - (\text{const}).$$

If we choose $|0\rangle$ to be the state that is annihilated by $a^s_\mathbf{p}$ and $\tilde{b}^s_\mathbf{p}$, then all excitations of $|0\rangle$ have positive energy.

What happened? To better understand this trick, let us abandon the field theory for a moment and consider a theory with a single pair of b and $b^\dagger$ operators obeying $\{b, b^\dagger\} = 1$ and $\{b, b\} = \{b^\dagger, b^\dagger\} = 0$. Choose a state $|0\rangle$ such that $b\,|0\rangle = 0$. Then $b^\dagger\,|0\rangle$ is a new state; call it $|1\rangle$. This state satisfies $b\,|1\rangle = |0\rangle$ and $b^\dagger\,|1\rangle = 0$. So b and $b^\dagger$ act on a Hilbert space of only two states, $|0\rangle$ and $|1\rangle$. We might say that $|0\rangle$ represents an "empty" state, and that $b^\dagger$ "fills" the state. But we could equally well call $|1\rangle$ the empty state and say that $b = \tilde{b}^\dagger$ fills it. The two descriptions are completely equivalent, until we specify some observable that allows us to distinguish the states physically. In our case the correct choice is to take the state of lower energy to be the empty one. And it is less confusing to put the dagger on the operator that creates positive energy. That is exactly what we have done.

Note, by the way, that since $(\tilde{b}^\dagger)^2 = 0$, the state cannot be filled twice. More generally, the anticommutation relations imply that any multiparticle state is antisymmetric under the interchange of two particles: $a^\dagger_\mathbf{p} a^\dagger_\mathbf{q}\,|0\rangle = -a^\dagger_\mathbf{q} a^\dagger_\mathbf{p}\,|0\rangle$. Thus we conclude that if the ladder operators obey *anticommutation* relations, the corresponding particles obey *Fermi-Dirac* statistics.

We have just shown that in order to insure that the vacuum has only positive-energy excitations, we must quantize the Dirac field with anticommutation relations; under these conditions the particles associated with the Dirac field obey Fermi-Dirac statistics. This conclusion is part of a more gen-