

# The equation of a Leptonic Monopole

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## Part I : General introduction.

### 1. History.

The hypothesis of separated magnetic poles is an old one. In the 2nd tome (devoted to Magnetism), of his famous *Treatise of Electricity and Magnetism* [1], Maxwell considered the existence of free magnetic charges as evident as the existence of electric charges. He based the theory of magnetism on this hypothesis and he reported that, as far back as 1785, Coulomb gave the *experimental proof* that the law of force of a magnetic charge is the same as the one of an electric charge : i.e. the « Coulomb law », in his experiments, he took as a magnetic charge, the extremity of a long, thin magnetic rod.

Given that our paper is not devoted to history or bibliography, we quote only three papers of this kind : [2], [3], [4] and later, we shall quote only papers useful for our purpose. Let us add that we remain in the framework of electrodynamics, without including other possible monopoles, as the one of Dirac (which is independent from his famous equation) or the one of t'Hoft and Polyakov.

Contrary to a certain tendency, supposing that a monopole must be heavy and bosonic, with strong interactions, (without clearly defined symmetry laws), *our monopole is based on symmetry and on Dirac's theory of the electron : it appears as a second slope of the electron and it is light, fermionic, with weak interactions*, and following features :

- In a different form, we find the Dirac formula between the electric and magnetic charges.
- This monopole is massless, but it admits a nonlinear massive generalisation.
- It appears as a magnetically excited neutrino, and it is linked to weak interactions.
- The gauge law from which derives the equation, automatically gives *in a quantum language* the symmetry laws of electromagnetism discovered by Pierre Curie.

### 2. Symmetry in classical form.

In his paper : "Symmetry in Physical Phenomena" [5]<sup>1</sup>, Pierre Curie put forward the constructive role of symmetry in physics. Generalizing the cristallographic groups, he defined the *invariance groups* of limited objects in  $\mathbb{R}^3$ , and applied them to electromagnetism, only starting from *experiment* without any use of Maxwell's equations. As a consequence he proved the *possibility* of "*free magnetic charges*"<sup>2</sup>[6].

The different symmetries of electric and magnetic charges are due to the fact that the electric field is a polar vector and the magnetic field is axial, which is proved on *experimental bases* [5]. For charges it was proved in the same way [7] but the P-invariance of the law of forces (2,1) gives a *formal* proof based on the variance of fields : the electric charge is scalar and magnetic charge pseudo-scalar :

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<sup>1</sup> A modern presentation of Curie's theory is given in [7] with a continuation concerning charges and time reversal.

<sup>2</sup> It is said in reference [2] that Curie "*suggests out of the blue that magnetic charge might exist*". Probably the authors have never read the paper of Curie in which this prediction is historically the *second* prediction of a physical fact based on symmetry. The *first one* already belonged to P. Curie (and his brother) : *have predicted in the same way piezoelectricity which was later observed by P. Curie*. It is no more "*out of the blue*", than the prediction of antimatter.

$$\begin{aligned}
P &: g \rightarrow g; x \\
T &: g \rightarrow g; \quad ; \mathbf{F} = g (\mathbf{H} - 1/c \mathbf{v} \times \mathbf{E}) \\
C &: g \rightarrow g;
\end{aligned}
\tag{2.1}$$

The scheme of classical symmetries for electromagnetic quantities is :

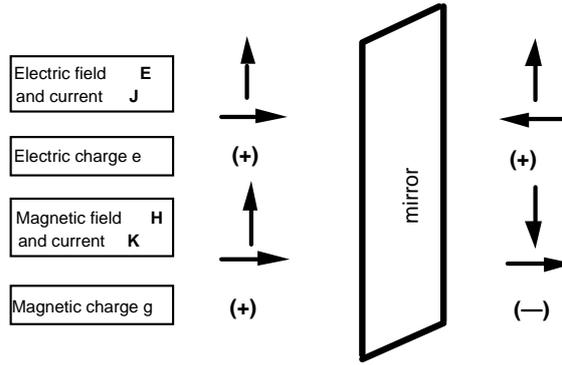


Fig. 1 Symmetry laws of electric and magnetic quantities

Owing to the P-variances of charges, the currents have the same P-variance as fields, because :

$$\mathbf{J} = e \mathbf{v}, \quad \mathbf{K} = g \mathbf{v} \tag{2.2}$$

Nevertheless, it is shocking to find that  $g$  is a pseudoscalar, because a physical constant has no tensorial variance :  $c$  does not vary as a velocity and  $h$  varies neither as an action nor as a kinetic moment. *It will be not so in quantum mechanics* : the magnetic charge will be a scalar, but the *pseudo-scalar* variance will be the property of a *charge operator*. Magnetic current will be an axial vector, but different from (2.2).

Thus, Fig.1 is formally true, but eq. (2.1) and (2.2) are not (in quantum mechanics). It is important because a classical objection against the hypothesis of magnetic poles is that they are purely formal because they can be eliminated from Maxwell's equations by a linear transformation which leaves the equations invariant. If  $(\mathbf{E}, \mathbf{H})$  are the fields,  $(\mathbf{J}, \mathbf{K})$  the electric and magnetic currents and  $(\rho, \mu)$  the electric and magnetic densities, the linear transformation is :

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}' \cos \gamma + \mathbf{H}' \sin \gamma; \quad \mathbf{H} = -\mathbf{E}' \sin \gamma + \mathbf{H}' \cos \gamma \\
\rho &= \rho' \cos \gamma + \mu' \sin \gamma; \quad \mu = -\rho' \sin \gamma + \mu' \cos \gamma \\
\mathbf{J} &= \mathbf{J}' \cos \gamma + \mathbf{K}' \sin \gamma; \quad \mathbf{K} = -\mathbf{J}' \sin \gamma + \mathbf{K}' \cos \gamma
\end{aligned}
\tag{2.3}$$

$\mathbf{K}$  may thus be eliminated by a good choice of the angle  $\gamma$ , but only if  $\mathbf{J}'$  and  $\mathbf{K}'$  are colinear, which is not true in our theory (see below) and this is the **unic known answer to this objection** [8].

### 3. The Birkeland-Poincaré effect.

In 1896, Birkeland introduced a magnet in a Crookes' tube (fig. 2) and was puzzled by a convergence of the cathodic beam [9]. Poincaré computed it as the action of a magnetic pole on an electric charge [10] [8].

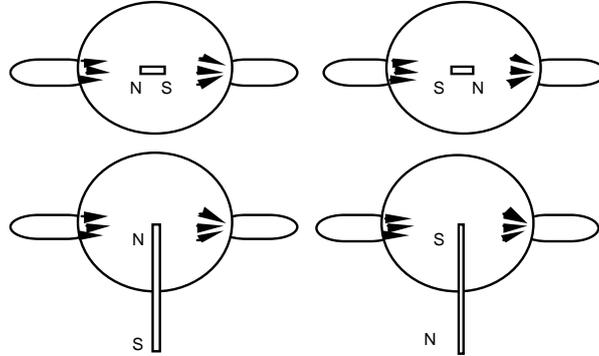


Fig. 2 The Birkeland-Poincaré effect. When a straight magnet is introduced in a Crookes' tube, the cathodic rays converge whatever the orientation of the magnet. Above : the cases considered by Birkeland ; below : two cases corresponding to the same description given by Poincaré.

Defining the Lorentz force of a magnetic charge  $g$ , Poincaré gave the equation :

$$\frac{d^2\mathbf{r}}{dt^2} = \lambda \frac{1}{r^3} \frac{d\mathbf{r}}{dt} \times \mathbf{r} ; \quad \lambda = \frac{eg}{mc} \quad (3.1)$$

where  $e$  and  $m$  are the electric charge and the mass of the cathodic particles<sup>3</sup>.

Using integrals of motion, Poincaré showed that  $\mathbf{r}$  follows a *geodesic line* of an axially symmetric cone (the *Poincaré cone*) and he proved the *focusing* effect observed by Birkeland.

**This is for us an important result** because we know that the Coulomb law is the same for electricity and magnetism so that **a magnetic monopole in a Coulomb electric field will obey the Poincaré equation** (only the  $\lambda$  constant varies). And we shall find later that **the Poincaré equation is the classical limit of the equation of our monopole** [21],[8] so that, **the fact that the Birkeland effect is predicted by the Poincaré equation is the first (indirect) experimental proof of the quantum equation that will be given below.**

Let us add two remarks :

1) *Poincaré found several integrals*, among which the total angular momentum  $\mathbf{J} = m\Lambda$  :

$$\Lambda = \mathbf{r} \times \frac{d\mathbf{r}}{dt} + \lambda \frac{\mathbf{r}}{r} \quad (3.2)$$

J.J. Thomson showed later that the second term is the electromagnetic momentum [11] [8].

2) The cone of Poincaré is enveloped by the *symmetry axis*  $\mathbf{r}$  (joining the electric and the magnetic charges), which rotates, with a constant angle  $\Theta'$ , around the constant *angular momentum*  $\mathbf{J} = m\Lambda$ , which is the definition of the *Poinsot cone* of a symmetric top : therefore, the angular symmetry of the system of an electric and a magnetic charge is the symmetry of a symmetric top [8], [13].

<sup>3</sup> It must be stressed that in that time, the electron was not yet discovered and most probably the concept of a free magnetic charge was not familiar to Poincaré : we have here an example of his intuition.

Introducing the following definition with two obvious properties :

$$\frac{d^2 \mathbf{r}}{dt^2} \cdot \mathbf{r} = \frac{d^2 \mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} = 0; \quad \Lambda \mathbf{r} = \lambda r \quad (3.3)$$

all that was said is summarized on the following figure :

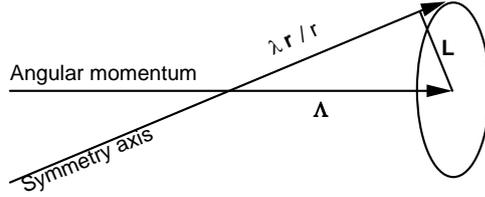


Fig. 3 The generation of the Poincaré (or Poinsot) cone and the decomposition of the total momentum.

Our equation for a magnetic monopole will give this cone in a quantum form.

#### 4. The electromagnetic potentials for a magnetic pole.

Let us write the Maxwell equations with currents ( $\mathbf{J}$ ,  $\mathbf{K}$ ) and charge densities ( $\rho$ ,  $\mu$ ) :

$$\begin{aligned} \text{curl } \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c} \mathbf{J}; \quad - \text{curl } \mathbf{E} - \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = \frac{4\pi}{c} \mathbf{K} \\ \text{div } \mathbf{E} &= 4\pi\rho; \quad \text{div } \mathbf{H} = 4\pi\mu \end{aligned} \quad (4.1)$$

Introducing the relativistic coordinates :

$$x^\alpha = \{x^1, x^2, x^3, x^4\} = \{x, y, z, ict\} \quad (4.2)$$

eq. (4.1) becomes, in covariant form, :

$$\partial^\beta F_{\alpha\beta} = \frac{4\pi}{c} J_\alpha; \quad J_\alpha = (\mathbf{J}, i\rho c); \quad \partial^\beta \bar{F}_{\alpha\beta} = \frac{4\pi}{c} K_\alpha; \quad iK_\alpha = (\mathbf{K}, i\mu c) \quad (4.3)$$

where «  $i$  » in front of  $K_\alpha$  is due to the axial character of  $K$  ; we have the duality :

$$\bar{\mathbf{F}}_{\alpha\beta} = \frac{i}{2} \varepsilon_{\alpha\beta\gamma\delta} \mathbf{F}^{\gamma\delta} \quad (\varepsilon_{\alpha\beta\gamma\delta} : \text{antisymmetric})$$

It is clear that we cannot define the field by a Lorentz *polar* potential only, because :

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \Rightarrow \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} \partial^\beta F^{\gamma\delta} = 0 \quad (4.4)$$

Therefore, we must introduce a new potential  $B_\alpha$  such that :

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} (\partial_\gamma B_\delta - \partial_\delta B_\gamma) \quad (4.5)$$

The right-hand terms in eq. (4.5) must have the same variance, so  $B_\alpha$  is a *pseudo-potential*, i.e. the dual of an antisymmetric tensor of rank three :

$$B_\alpha = \frac{1}{3!} \varepsilon_{\alpha\beta\gamma\delta} C^{\beta\gamma\delta} \quad (4.6)$$

In terms of ordinary coordinates, we have :

$$A_\alpha = (\mathbf{A} \cdot iV); \quad iB_\alpha = (\mathbf{B} \cdot iW) \quad (4.7)$$

where  $\mathbf{B}$  is an axial vector. The fields are defined as :

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \text{curl } \mathbf{B}; \quad \mathbf{H} = \text{curl } \mathbf{A} + \nabla W + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (4.8)$$

We shall consider monopoles with a magnetic charge only. Thus, formulae (4.5) and (4.8) become :

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{s} = \int_\Sigma \text{curl } \mathbf{B} \cdot d\mathbf{s} = \int_\Lambda \mathbf{B} \cdot d\mathbf{l} = e \int_\Sigma \frac{\mathbf{r}}{r^3} \cdot d\mathbf{s} = e \int_\Sigma d\Omega \quad (4.9)$$

Given at first by de Broglie [14] they were related to the monopole by Cabibbo and Ferrari [15].

## 5. Dirac strings :

In 1931, Dirac raised the problem of the motion of an electric charge around a fixed monopole or conversely [16]. We chose the motion of a magnetic charge  $g$  around an electric charge  $e$ .  $\mathbf{E}$  is defined by :

$$\text{curl } \mathbf{B} = e \frac{\mathbf{r}}{r^3} \quad (5.1)$$

There cannot be a continuous uniform solution  $\mathbf{B}$  because if we consider a surface  $\Sigma$  bounded by a loop  $\Lambda$ , we find according to Stokes' theorem ( $d\mathbf{s}$ ,  $d\mathbf{l}$ ,  $d\Omega$  = element s of surface, length and solid angle) :

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{s} = \int_\Sigma \text{curl } \mathbf{B} \cdot d\mathbf{s} = \int_\Lambda \mathbf{B} \cdot d\mathbf{l} = e \int_\Sigma \frac{\mathbf{r}}{r^3} \cdot d\mathbf{s} = e \int_\Sigma d\Omega \quad (5.2)$$

With a continuous potential  $\mathbf{B}$ , the first integral vanishes and so **there must be a singular line around which the loop shrinks : the Dirac string**. Saving the uniformity of wave functions, Dirac found his **famous relation** between electric and magnetic elementary charges [16], [8] :

$$\frac{eg}{hc} = \frac{n}{2} \quad (5.3)$$

We shall not give Dirac's proof that may be found in quoted references [16], [8] because we shall give in the Part II a proof based on our equation [8]. We shall note here only two points :

- In the Dirac reasoning, the **string plays a central role**. In our theory, **the string will be as to say rubbed out by an argument of isotropy**.
- Dirac's choice of potentials corresponds to the following solution of (5,1) :

$$B_x = \frac{e}{r} \frac{-y}{r+z}, B_y = \frac{e}{r} \frac{x}{r+z}, B_z = 0 \quad \left( r = \sqrt{x^2 + y^2 + z^2} \right) \quad (5.4)$$

This solution is **not symmetry definite** which makes calculations more difficult. We shall chose, in the following, another gauge for  $\mathbf{B}$  that gives as a solution a true **pseudo vectorial potential**, in accordance with the symmetry of the problem, owing to which the problem is strongly simplified:

$$B_x = \frac{e}{r} \frac{yz}{x^2 + y^2}, B_y = \frac{e}{r} \frac{-xz}{x^2 + y^2}, B_z = 0 \quad \left( r = \sqrt{x^2 + y^2 + z^2} \right) \quad (5.5)$$

## Part II : The leptonic monopole.

### 6. Symmetry in quantum form.

Now, we shall find the wave equation of the magnetic monopole. It will be shown that the Dirac equation for the electron admits not only one **local gauge** but two, and only two. The first invariance corresponds to an electric charge the second to a magnetic monopole. The new spinorial equations describe, in quantum terms, the Curie symmetry laws. New symmetries appear, only in quantum mechanics, without any classical equivalence : especially the correct definition of an antimonopole.

#### a) The two gauges of Dirac's equation.

Let us write the Dirac equation without external field :

$$\gamma_\mu \partial_\mu \Psi + \frac{m_0 c}{\hbar} \Psi = 0 \quad (6.1)$$

$x_\mu = \{x_k ; i c t\}$  and  $\gamma_\mu$  are matrices defined in terms of Pauli matrices  $s_k$  as :

$$\gamma_k = i \begin{pmatrix} 0 & s_k \\ -s_k & 0 \end{pmatrix}; k = 1, 2, 3; \gamma_4 = \begin{pmatrix} I & O \\ 0 & -I \end{pmatrix}; \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (6.2)$$

Consider a global gauge transformation where  $\Gamma$  is a Hermitian matrix and  $\theta$  a constant phase :

$$\Psi \rightarrow e^{i\Gamma\theta} \Psi \quad (6.3)$$

Eq. (6.1) becomes :

$$\left( \gamma_\mu e^{i\Gamma\theta} \gamma_\mu \right) \gamma_\mu \partial_\mu \Psi + \frac{m_0 c}{\hbar} e^{i\Gamma\theta} \Psi = 0 \quad (6.4)$$

Let us now developpe  $\Gamma$  on the Clifford algebra basis built on the  $\gamma_\mu$  matrices :

$$\Gamma = \sum_{N=1}^{16} a_N \Gamma_N; \Gamma_N = \left\{ \gamma_\mu, \gamma_{[\mu} \gamma_{\nu]}, \gamma_{[\lambda} \gamma_\mu \gamma_{\nu]} \gamma_5 \right\} \quad (6.5)$$

The commutation rules of  $\gamma_\mu$  give the following relations for  $\Gamma_N$  [17] (the sign varies with  $\mu$  and N) :

$$\gamma_\mu \Gamma_N \gamma_\mu = \pm \Gamma_N \quad (6.6)$$

Hence we find from eq. (6.4) :

$$\gamma_\mu e^{i\Gamma\theta} \gamma_\mu = \exp\left(i\theta \sum_{N=1}^{16} \pm a_N \gamma_\mu \Gamma_N \gamma_\mu\right) = \exp\left(i\theta \sum_{N=1}^{16} \pm a_N \Gamma_N\right) \quad (6.7)$$

The gauge invariance of eq. (6.1) needs that the factor  $\gamma_\mu e^{i\Gamma\theta} \gamma_\mu$  in (6.4) does not depend on  $\mu$  which is possible only if  $\Gamma$  commutes or anticommutes with all the  $\gamma_\mu$  i.e. if  $\Gamma = \mathbf{I}$  or  $\Gamma = \gamma_5$ . The first case gives the *ordinary phase invariance* and the *conservation of electricity*. The second term will be called *chiral invariance* and will give the *conservation of magnetism*

$$if \Gamma = \mathbf{I}, \Psi \rightarrow e^{i\theta} \Psi ; if \Gamma = \gamma_5, \Psi \rightarrow e^{i\gamma_5\theta} \Psi \quad (6.8)$$

These gauges are quite different because the first one is valid for every value of  $m_0$  in eq. (6.1), so that the conservation of electricity is universal in quantum mechanics ; the second one (which seems to have appeared in the Dirac theory, for the first time in [18], [20]) is valid only for  $m_0 = 0$  because of the anticommutation of  $\gamma_5$  and  $\gamma_\mu$ , so that *the conservation of magnetism is weaker than conservation of electricity*. Nevertheless, there is a symmetry between the two gauges.

#### b) The Dirac tensors and the magic angle A of Yvon-Takabayasi.

In the Clifford algebra basis (6.5), the Dirac spinor defines 16 tensorial quantities. A scalar, a polar vector, an antisymmetric tensor of rank two, an antisymmetric tensor of rank three (an axial vector) and an antisymmetric tensor of rank four (a pseudo-scalar) :

$$\begin{aligned} \Omega_1 &= \bar{\Psi} \Psi ; J_\mu = i \bar{\Psi} \gamma_\mu \Psi ; M_{\mu\nu} = i \bar{\Psi} \gamma_\mu \gamma_\nu \Psi ; \Sigma_\mu = i \bar{\Psi} \gamma_\mu \gamma_5 \Psi ; \Omega_2 = -i \bar{\Psi} \gamma_5 \Psi \\ (\bar{\Psi} &= \Psi^\dagger \gamma_4 ; \Psi^\dagger = \Psi h.c.) \end{aligned} \quad (6.9)$$

When  $\Omega_1$  and  $\Omega_2$  do not simultaneously vanish, the Dirac spinor may be written as [18], [19], [21] :

$$\Psi = \rho e^{i\gamma_5 A} U \Psi_0 \quad (6.10)$$

( $\rho$ =amplitude,  $A$  = *pseudo-scalar angle* of Yvon-Takabayasi,  $U$  = general Lorentz transformation,  $\Psi_0$  = constant spinor):

$$\rho = \sqrt{\Omega_1^2 + \Omega_2^2} ; A = \text{Arctg} \frac{\Omega_2}{\Omega_1} \quad (6.11)$$

The proper rotation Euler angle  $\varphi$  (defined in U) is a *scalar phase*  $\tilde{\varphi}/2$  of the spinor  $\Psi$ , which is conjugated (by a *classical Poisson bracket*) to the fourth component of the *polar vector*  $J_\mu$  ; the *pseudo-scalar angle*  $A$  is conjugated to the fourth component of the *axial vector*  $\Sigma_\mu$  [ [18], [19], [21] :

$$\left[ \frac{\varphi}{2}, J_4 \right] = \delta(\mathbf{r} - \mathbf{r}') ; \left[ \frac{A}{2}, \Sigma_4 \right] = \delta(\mathbf{r} - \mathbf{r}') \quad (6.12)$$

In Dirac's theory of the electron,  $J_4$  is a density of *electricity* associated with the phase invariance ; the spatial part  $\mathbf{J}$  of  $J_\mu$  is a density of electric current.  $\Sigma_4$  is a density associated in the same way to the chiral invariance and it will be shown that the space part of  $\Sigma_\mu$  is a density of current. There will be densities of charge and current of *magnetism*. The differences between the two gauges are that : 1)  $J_\mu$  is *polar* and  $\Sigma_\mu$  *axial*, 2)  $J_\mu$  is *time-like* and  $\Sigma_\mu$  is *space-like* because of the Darwin - de Broglie equalities :

$$-\mathbf{J}_\mu \mathbf{J}_\mu = \Sigma_\mu \Sigma_\mu = \Omega_1^2 + \Omega_2^2 ; \quad \mathbf{J}_\mu \Sigma_\mu = 0 \quad (6.13)$$

It is because  $J_\mu$  is *time-like*, that it may be interpreted as a current of electricity and probability. At a first glance, a space-like magnetic current  $\Sigma_\mu$  seems to be unacceptable but we shall see that it is not so.

### c) PTC symmetries of the angle A.

It may be proved [7] that the correct, in the sense of Pierre Curie laws, **PTC** transformations in Dirac's theory are the following (we see that **P** is a Racah transformation, but **T** is not) :

$$\begin{aligned} \mathbf{P} : \Psi &\rightarrow \gamma_4 \Psi \\ \mathbf{T} : \Psi &\rightarrow -i \gamma_3 \gamma_1 \Psi, \quad (e \rightarrow -e) \\ \mathbf{C} : \Psi &\rightarrow \gamma_2 \Psi^*, \quad (e \rightarrow -e) \end{aligned} \quad (6.14)$$

With the definitions (6.9), this implies :

$$\begin{aligned} \mathbf{P} : \Omega_1 &\rightarrow \Omega_1 ; \Omega_2 \rightarrow -\Omega_2 \\ \mathbf{T} : \Omega_1 &\rightarrow \Omega_1 ; \Omega_2 \rightarrow -\Omega_2 \\ \mathbf{C} : \Omega_1 &\rightarrow -\Omega_1 ; \Omega_2 \rightarrow -\Omega_2 \end{aligned} \quad (6.15)$$

(6.11) shows that **A is a relativistic pseudo-invariant and that it is PTC invariant**. We can give a more geometrical interpretation of the chiral gauge transformation, owing to (6.11) and the equalities [8] :

$$\Omega_1 = \rho \cos A ; \quad \Omega_2 = \rho \sin A \quad (6.16)$$

Now, consider a chiral gauge transformation, slightly modified with respect to the eq. (6,10) :

$$\Psi = e^{i \gamma_3 \theta / 2} \Psi \quad (6.17)$$

Using definition (6,11) of  $\Omega_1$  and  $\Omega_2$ , we get from (6,16) :

$$\begin{pmatrix} \Omega'_1 \\ \Omega'_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \quad (6.18)$$

**The chiral gauge transformation is thus a rotation in the plane  $\{\Omega_1, \Omega_2\}$**  (while the rotation of the spinor in this plane is  $\tilde{\theta}/2$ ). Making use of (6.1), we verify that  $\theta$  represents a phase shift of the angle A :

$$A' = A + \theta \quad (6.19)$$

Naturally,  $\theta$  is a *relativistic pseudo-scalar*, like A.

**d) The wave equation.**

It is wellknown that the *local* gauge deduced from the first gauge (6,9) gives the minimal electric coupling and the Dirac equation of the electron. Now, consider the Dirac equation with  $m_0 = 0$  :

$$\gamma_\mu \partial_\mu \Psi = 0 \quad (6.20)$$

and the chiral gauge (6.8), with a pseudo-scalar phase  $\phi$  and physical coefficients, defining :

$$\Psi \rightarrow \exp \left( i \frac{g}{\hbar c} \gamma_5 \phi \right) \Psi ; B_\mu \rightarrow B_\mu + i \partial_\mu \phi ; G = g \gamma_5 \quad (6.21)$$

Here,  $g$  is a *scalar* magnetic charge : the pseudo-scalar character of magnetism is related to a *pseudo-scalar magnetic charge operator*  $G$  which is at the origin of all the differences between the classical and the quantum theory of magnetic monopoles.

As  $\phi$  is a pseudo-scalar, the electromagnetic potential cannot be the Lorentz polar vector  $\mathbf{A}_\mu$ , but the *axial potential*  $\mathbf{B}_\mu$  defined by (4.6), (4.7) and which have the variance of  $\partial_\mu \phi$ . The covariant derivatives are (the absence of  $i$  in front of  $g$  is due to the axiality of  $\mathbf{B}_\mu$ ) :

$$\nabla_\mu = \partial_\mu - \frac{g}{\hbar c} \gamma_5 \mathbf{B}_\mu \quad (6.22)$$

**The equation of the magnetic monopole** is thus [21], [22]:

$$\gamma_\mu \left( \partial_\mu - \frac{g}{\hbar c} \gamma_5 B_\mu \right) \Psi = 0 \quad (6.23)$$

The justification of this equation will be given by its symmetry and other properties.

**7. Symmetries of the wave equation.****a) Gauge invariance.**

By definition, eq. (6.23) is invariant with respect to the chiral gauge transformation (6.21). This entails the conservation of the *axial current* that plays the role of a magnetic current :

$$\partial_\mu K_\mu = 0 ; K_\mu = g \Sigma_\mu = i g \bar{\Psi} \gamma_\mu \gamma_5 \Psi \quad (7.1)$$

According to (6,13), the magnetic current cannot be colinear to the electric current, which makes impossible to destroy  $(\mathbf{K}_\mu, \rho)$  in (2.3)).  $\mathbf{K}_\mu$  is pseudo-tensorial, as it was predicted by Curie. Its space-like character will become clear a little further. This expression for the magnetic current was suggested by Salam [23] for reasons of symmetry, but here, it is *deduced* from a wave equation and a gauge condition.

**b) CPT.**

It is easy to prove that the wave equation (6.23) is C, P and T invariant, i.e. invariant under the transformations [7]<sup>4</sup>

$$\begin{aligned} P &: g \rightarrow g ; x_k \rightarrow -x_k ; x_4 \rightarrow x_4 ; B_k \rightarrow B_k ; B_4 \rightarrow -B_4 ; \Psi \rightarrow \gamma_4 \Psi \\ T &: g \rightarrow g ; x_k \rightarrow x_k ; x_4 \rightarrow -x_4 ; B_k \rightarrow -B_k ; B_4 \rightarrow B_4 ; \Psi \rightarrow -i \gamma_3 \gamma_1 \Psi^* \\ C &: g \rightarrow g ; \Psi \rightarrow \gamma_2 \Psi^* \end{aligned} \quad (7.2)$$

In this formulae, the most important point is that *the charge conjugation does not change the sign of the magnetic constant of charge*. In the next section, we shall see what exactly charge conjugation means, but we can already assert that two conjugated monopoles have the same charge constant and that two monopoles with opposite charges are not charge conjugated : *changing g in -g in the eq. (6.23) means something else : the change of the angle of the Poincaré cone*.

We cannot create or annihilate pairs of monopoles with opposite charges *g and -g*, in the same way as pairs of electric charges. The properties of charge conjugation of eq. (6.23) show that there is no danger of an infinite polarization of vacuum which could occur from the zero mass of our monopole and it shows that *one cannot invoke the hypothesis of great masses* to explain the rarity of monopoles or, at least, the difficulty to observe them. The fact that chiral invariance and conservation of magnetism are easily broken shows that, more probably, monopoles are abundant in nature and that the difficulty of the isolation of one of them is not a question of energy.

## 8. Weyl's representation. Two-component theory.

The Weyl representation diagonalizes  $\gamma_5$  and the charge operator G. The transformation is :

$$\Psi \rightarrow U\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} ; U = U^{-1} \frac{1}{\sqrt{2}} (\gamma_4 + \gamma_5) \quad (8.1)$$

where  $\xi$  and  $\eta$  are two-component spinors ; and we have :

$$UGU^{-1} = Ug\gamma_5U^{-1} = g\gamma_4 = \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix} \quad (8.2)$$

Applying eq. (8.2) to  $\psi$  given by (8.1), we see that  $\xi$  and  $\eta$  are eigenstates of G, corresponding to the eigenvalues  $g$  and  $-g$  :

$$UGU^{-1} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = g \begin{pmatrix} \xi \\ 0 \end{pmatrix} ; UGU^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix} = -g \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad (8.3)$$

Owing to (8.1) and (4.7), *the equation (6.23) splits into a pair of uncoupled two component equations in  $\xi$  and  $\eta$  corresponding to opposite eigenvalues of the charge operator G [21], [22] :*

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<sup>4</sup> In (7.2), there is a contradiction concerning T with some preceding papers of G.L. who first extended to magnetism the common formula of Racah :  $e \rightarrow e, \Psi \rightarrow -i \gamma_1 \gamma_2 \gamma_3 \Psi$ . He later discovered [7] that this formula contradicts certain laws of electromagnetism and adopted the correct law  $e \rightarrow -e, \Psi \rightarrow -i \gamma_3 \gamma_1 \Psi^*$  in the electric case and  $g \rightarrow g, \Psi \rightarrow -i \gamma_3 \gamma_1 \Psi^*$  in the magnetic case.

$$\begin{aligned} \left[ \frac{1}{c} \frac{\partial}{\partial t} - \mathbf{s} \cdot \nabla - i \frac{g}{\hbar c} (W + \mathbf{s} \cdot \mathbf{B}) \right] \xi &= 0 \\ \left[ \frac{1}{c} \frac{\partial}{\partial t} + \mathbf{s} \cdot \nabla + i \frac{g}{\hbar c} (W - \mathbf{s} \cdot \mathbf{B}) \right] \eta &= 0 \end{aligned} \quad (8.4)$$

They **exchange** between themselves by **P** and **T** transformations and are **PTC** invariant.

$$\begin{aligned} P : g &\rightarrow g ; x_k \rightarrow -x_k ; t \rightarrow t ; B_k \rightarrow B_k ; W \rightarrow -W ; \xi \leftrightarrow \eta \\ T : g &\rightarrow g ; x_k \rightarrow x_k ; t \rightarrow -t ; B_k \rightarrow -B_k ; W \rightarrow W ; \xi \rightarrow s_2 \xi^* ; \eta \rightarrow s_2 \eta^* \\ C : g &\rightarrow g ; \xi \rightarrow -i s_2 \eta^* ; \eta \rightarrow i s_2 \xi^* \end{aligned} \quad (8.5)$$

They describe two *charge conjugated* particles — a *monopole* and an *antimonopole* — with the same charge constant but *opposite helicities*, defined by the operator **G**, which shows that **our monopole is a magnetically excited neutrino** : (8.4) reduces to the neutrino two-component equations if  $g = 0$  [22].

The eq. (8.4) are invariant under a gauge transformation (with opposite signs of  $\phi$  for  $\xi$  and  $\eta$ ) :

$$\xi \rightarrow \exp \left( i \frac{g}{\hbar c} \phi \right) \xi ; \eta \rightarrow \exp \left( -i \frac{g}{\hbar c} \phi \right) \eta ; W \rightarrow W + \frac{1}{c} \frac{\partial \phi}{\partial t} ; \mathbf{B} \rightarrow \mathbf{B} - \nabla \phi \quad (8.6)$$

## 9. Chiral currents.

The gauge law (8.6) entails, respectively, for the equations (8.4), the *conservation of chiral currents* :

$$\frac{1}{c} \frac{\partial (\xi^+ \xi)}{\partial t} - \nabla \xi^+ \mathbf{s} \xi = 0 ; \frac{1}{c} \frac{\partial (\eta^+ \eta)}{\partial t} + \nabla \eta^+ \mathbf{s} \eta = 0 \quad (9.1)$$

These currents are thus defined as :

$$X_\mu = (\xi^+ \xi \quad -\xi^+ \mathbf{s} \xi) ; Y_\mu = (\eta^+ \eta \quad \eta^+ \mathbf{s} \eta) \quad (9.2)$$

Its is easy to prove that :

$$X_\mu X_\mu = 0 ; Y_\mu Y_\mu = 0 ; \mathbf{P} \Rightarrow X_\mu \leftrightarrow Y_\mu \quad (9.3)$$

So, they are *isotropic* and they exchange between themselves by parity : they are *chiral currents*. Owing to (8.1), we find the following decomposition of the polar and axial vectors defined in (6.10) :

$$J_\mu = X_\mu + Y_\mu ; \Sigma_\mu = X_\mu - Y_\mu \quad (9.5)$$

One can **consider the chiral isotropic currents  $X_\mu$  and  $Y_\mu$  as the fundamental currents and to define the electric and magnetic currents as their sum and their difference** (with suitable charge factors). Identities (6.13) are now easily proved, using (6.11) and (8.1). We find :

$$\Omega_1 = \xi^+ \eta + \eta^+ \xi ; \Omega_2 = i (\xi^+ \eta - \eta^+ \xi) ; \rho^2 = 4 (\xi^+ \eta) (\eta^+ \xi) \quad (9.6)$$

The fact that  $J_\mu$  (*resp.*  $\Sigma_\mu$ ) must be time-like and conversely  $\Sigma_\mu$  (*resp.*  $J_\mu$ ) be space-like is a trivial property of the addition of isotropic vectors. But why *precisely*  $J_\mu$  is space-like is due to (6.13) and

(9.6). **Our magnetic current  $\mathbf{K}_\mu = \mathbf{g}\Sigma_\mu$  may be space-like** because the true magnetic currents are the chiral currents  $gX_\mu$  and  $gY_\mu$  whereas  $K_\mu$  is only their difference, which has no definite type.

### 10. The geometrical optics approximation and the Poincaré equation

Now we must verify that we find the correct Poincaré equation and the Birkeland effect. Let us introduce in the first equation (8.4) the following expression of the spinor  $\xi$  :

$$\xi = a e^{iS/\hbar} \quad (10.1)$$

where  $a$  is a two-component spinor and  $S$  a phase. At zeroth order in  $\hbar$ , we have :

$$\left[ \frac{1}{c} \left( \frac{\partial}{\partial t} - gW \right) - \left( \nabla S + \frac{\mathbf{g}}{c} \mathbf{B} \right) \cdot \mathbf{s} \right] a = 0 \quad (10.2)$$

This is an homogeneous system with respect to  $a$ . A necessary condition for a non trivial solution is :

$$\frac{1}{c^2} \left( \frac{\partial}{\partial t} - gW \right)^2 - \left( \nabla S + \frac{\mathbf{g}}{c} \mathbf{B} \right)^2 = 0 \quad (10.3)$$

This is nothing but a relativistic Jacobi equation with zero mass and we may define the kinetic energy, the impulse and the linear Lagrange momentum :

$$E = -\frac{\partial S}{\partial t} + gW; \quad \mathbf{p} = \nabla S + \frac{\mathbf{g}}{c} \mathbf{B}; \quad \mathbf{P} = \nabla S \quad (10.4)$$

The Hamiltonian function will be equal to :

$$H = c \sqrt{\left( \mathbf{P} + \frac{\mathbf{g}}{c} \mathbf{B} \right)^2} - gW \quad (10.5)$$

and a classical calculation gives as equation of motion :

$$\frac{d\mathbf{p}}{dt} = g \left( \nabla W + \frac{\partial \mathbf{B}}{\partial t} \right) - \frac{\mathbf{g}}{c} \mathbf{v} \times \text{curl } \mathbf{B} \quad (10.6)$$

The eq. (4.9) gives the classical form :

$$\frac{d\mathbf{p}}{dt} = g \left( \mathbf{H} - \frac{1}{c} \mathbf{v} \times E \right) \quad (10.7)$$

But we must not forget that the mass of our particle is equal to zero, so that  $\mathbf{v}$  is the velocity of light and we cannot write :  $\mathbf{p} = m \mathbf{v}$ . But the equality :  $\mathbf{p} = \frac{E}{c^2} \mathbf{v}$  still holds when the energy  $E$  is a constant, which will be the case in a coulombian electric field. We then have :

$$\frac{d\mathbf{p}}{dt} = -\lambda \frac{1}{\mathbf{r}^3} \mathbf{p} \times \mathbf{r} ; \quad \lambda = \frac{ecg}{E} \quad (10.8)$$

**This is the Poincaré equation** (3.1) with a minus sign because we have chosen the left monopole. There is a problem with the right monopole which cannot be deduced from the former by changing the sign of charge but by changing the sign of *the phase of the wave*, with the same magnetic charge [8].

### 11. The monopole in an electric central field. Angular eigenfunctions. Dirac's condition.

We must introduce  $W = 0$  and the expressions (5.5) of  $\mathbf{B}$  in (8.4). First of all a quite simple calculation gives the following integrals of motion, respectively for the monopole and the antimonopole or, equivalently for the left and the right monopole [22] :

$$\begin{aligned} \mathbf{J}_\xi &= \hbar \left[ \mathbf{r} \times (-\mathbf{i} \nabla + \delta \mathbf{B}) + \delta \frac{\mathbf{r}}{r} + \frac{1}{2} \mathbf{s} \right] \\ \mathbf{J}_\eta &= \hbar \left[ \mathbf{r} \times (-\mathbf{i} \nabla - \delta \mathbf{B}) - \delta \frac{\mathbf{r}}{r} + \frac{1}{2} \mathbf{s} \right]; \left( \delta = \frac{e g}{\hbar c}; \mathbf{B} = e B \right) \end{aligned} \quad (11.1)$$

$D$  is the *Dirac number* that already appeared in the *Dirac condition* (5.3).  $\mathbf{J}_\xi$  and  $\mathbf{J}_\eta$  only differ by the sign of  $D$  (i.e. by the sign of the eigenvalues of the charge operator). We only consider the plus sign, which corresponds to the first equation (8.4) (the left monopole), and we shall drop the  $\xi$  index. It is easily shown that the components of  $\mathbf{J}$  obey the relations of an angular momentum :

$$[J_2, J_3] = i \hbar J_1; [J_3, J_1] = i \hbar J_2; [J_1, J_2] = i \hbar J_3 \quad (11.2)$$

Now, if we write  $\mathbf{J}$  as :

$$\mathbf{J} = \left[ \Lambda + \frac{1}{2} \mathbf{s} \right]; \Lambda = \mathbf{r} \times (-\mathbf{i} \nabla + \delta \mathbf{B}) + \delta \frac{\mathbf{r}}{r} \quad (11.3)$$

we recognize that  $\hbar \Lambda$  is *the quantum form of the Poincaré integral* (3.2).  $\mathbf{J}$  is the sum of this integral and of the spin operator :  $\mathbf{J}$  is the *total quantum angular momentum* of the monopole in an electric coulombian field, the exact analogue of the corresponding classical quantity. Of course, the components of  $\hbar \Lambda$  obey the same relations (11.3) as the components of  $\mathbf{J}$ .

Translating  $\mathbf{B}$  given by (5.5) in terms of polar angles, we find, from the definition (11.3) :

$$\begin{aligned} \Lambda^+ &= \Lambda_1 + i \Lambda_2 = e^{i\varphi} \left( i \cot \theta \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \theta} + \frac{\delta}{s i n \theta} \right) \\ \Lambda^- &= \Lambda_1 - i \Lambda_2 = e^{-i\varphi} \left( i \cot \theta \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \theta} + \frac{\delta}{s i n \theta} \right) \\ \Lambda_3 &= -i \frac{\partial}{\partial \varphi} \end{aligned} \quad (11.4)$$

It is important for the calculation and its interpretation that, owing to our choice of gauge (5.5), there is *no additional term* in  $\Lambda_3$  as it occurred with the Dirac solution [24], [25]. Now, we look for the eigenstates  $Z(\theta, \varphi)$  of  $(\Lambda)^2$  and  $\Lambda_3$  knowing from (11.2), that the eigenvalue equations are :

$$\begin{aligned} (\Lambda^2)Z &= j(j+1)Z; \Lambda_3 Z = mZ; \\ j &= 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots; m = -j, -j+1, \dots, j-1, j \end{aligned} \quad (11.5)$$

To simplify the calculation, let us introduce an angle  $\chi$  the meaning of which will soon appear and a function  $D(\theta, \varphi, \chi)$ :

$$D(\theta, \varphi, \chi) = e^{i\delta\chi} Z(\theta, \varphi) \quad (11.6)$$

This function is an eigenstate of operators  $R_k$  that can be easily derived from (11.4) :

$$\begin{aligned} R^+ &= R_1 + iR_2 = e^{i\varphi} \left( i \cot \theta \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \chi} \right) \\ R^- &= R_1 - iR_2 = e^{-i\varphi} \left( i \cot \theta \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \chi} \right) \\ R_3 &= -i \frac{\partial}{\partial \varphi} \end{aligned} \quad (11.7)$$

Obviously, the eigenvalues are the same as those of  $Z$  :

$$(R^2)D(\theta, \varphi, \chi) = j(j+1)D(\theta, \varphi, \chi); R_3 Z = mD(\theta, \varphi, \chi) \quad (11.8)$$

The  $R_k$  are the *infinitesimal operators of the rotation group* written in the fixed referential, and  $\theta, \varphi, \dots, \chi$  are the nutation, the precession and the proper rotation. The role of the rotation group is evident because of the *spherical symmetry* of the system constituted by a monopole in a central electric field.

**Our eigenfunction problem is trivially solved by the hypothesis of continuity on the rotation group** instead of the cumbersome calculations of "monopole harmonics" [24], [25] (that actually don't exist !). Owing this continuity, the effects of the Dirac string are « rubbed out » as was already said.

Under this assumption of *continuity on the rotation group*, we find that the angular eigenfunctions are the *generalized spherical functions*, i.e. the matrix elements of the irreducible unitary representations of the rotation group [8], [21], [22], [26], [27]. And they also are the eigenfunctions of the spherical top which was quoted by Tamm [28] as a coincidence, but it becomes evident here as a consequence of the analogy between the system of a monopole in a central field and the *angular motion of a symmetric top*.

The eigenstates of  $R^2$  and  $R_3$  are given by the group theory and the end of the calculation may be found in [8], [22]. But, the most important point already appears on the formula (11,6). The eigenstates  $D(\theta, \varphi, \chi)$  must be identified with the matrix elements  $D_j^{m', m}(\theta, \varphi, \chi)$  of the unitary representations of the rotation group. So appear the eigenvalues  $j, m, m'$  :

$$j, m, m', \text{ with } : j = 0, 1, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots; m, m' = -j, -j+1, \dots, j-1, j \quad (11,9)$$

$j$  are the values of the *total angular momentum* and  $m'$  is the **projection of this momentum on the symmetry axis of the system, which joins the monopole and the coulombian center**.

But **this number  $m'$  is the number  $\delta$  in factor of  $\chi$  of the exponent in (11,6)**. So, we have :  $\delta = m'$  and we know from (11,2) that  $\delta$  is the Dirac Number and so we have :

$$\delta = m' = \frac{e\mathcal{G}}{\hbar c} = -j, -j + 1, \dots, j - 1, j \quad (11,10)$$

Thus, we find the Dirac formula in another form :

- 1) The proof is based on a model that gives a physical sense to the numbers  $n$  in (5,3).
- 2) It appears that this number is limited by the quantum state of the « top », which shows that the formula is not so universal as it seems : probably related to the interaction between two charges. Actually, it is absent from all the other problems solved owing the equations (6.23) and (8,4).

Now, the normalized angular eigenfunctions take the form [8], [22] :

$$Z_j^{m',m}(\theta, \varphi) = \sqrt{2j+1} D_j^{m',m}(\theta, \varphi, 0) (i)^{m'-m} \quad (11.11)$$

The proper rotation angle  $\chi$  disappears because the monopole was implicitly supposed to be *punctual*, contrary to the symmetric top. But there is a projection, different from zero, of the orbital angular momentum on the symmetry axis, due to the chirality of the magnetic charge.

We shall not give here the calculation of the radial part of eigenfunctions tyhat may be find in [8], [22].

## 12. A nonlinear massive monopole.

Until now we had a *massless* linear monopole (6.23) (non Abelian gauge). There are *nonlinear chiral invariant* (i.e. independent from A) generaliations. The most general mass is a function of  $F(\rho)$  : [ $\rho$  given by (6.11)]. In Weyl's representation [7], [8], [22] we find the lagrangian and equations:

$$\begin{aligned} L = & \frac{\hbar c}{i} \left\{ \xi^\dagger \left( \frac{1}{2} \frac{1}{c} [\partial_t] - \frac{\mathcal{G}}{\hbar c} W \right) \xi - \xi^\dagger \mathbf{s} \cdot \left( \frac{1}{2} [\nabla] + \frac{\mathcal{G}}{\hbar c} \mathbf{B} \right) \xi \right\} + \\ & + \frac{\hbar c}{i} \left\{ \eta^\dagger \left( \frac{1}{2} \frac{1}{c} [\partial_t] + \frac{\mathcal{G}}{\hbar c} W \right) \eta + \eta^\dagger \mathbf{s} \cdot \left( \frac{1}{2} [\nabla] - \frac{\mathcal{G}}{\hbar c} W \right) \eta \right\} + \hbar c F(\rho) \end{aligned} \quad (12.1)$$

$$\begin{aligned} \frac{1}{c} \partial_t \xi - \mathbf{s} \cdot \nabla \xi - i \frac{\mathcal{G}}{\hbar c} (W + \mathbf{s} \cdot \mathbf{B}) \xi + i \kappa(\rho) \sqrt{\frac{\eta^\dagger \xi}{\xi^\dagger \eta}} \eta &= 0 \\ \frac{1}{c} \partial_t \eta + \mathbf{s} \cdot \nabla \eta + i \frac{\mathcal{G}}{\hbar c} (W - \mathbf{s} \cdot \mathbf{B}) \eta + i \kappa(\rho) \sqrt{\frac{\xi^\dagger \eta}{\eta^\dagger \xi}} \xi &= 0 ; \quad \left( \kappa(\rho) = \frac{d F(\rho)}{d \rho} \right) \end{aligned} \quad (12.2)$$

The equations are chiral invariant, the magnetic current (7.1) is conserved and, owing to (7.2), the equations are **P** and **T** invariant but not **C** invariant [7]. In general, eqs. (12.2) are coupled, contrary to (8.4) but this coupling is not strong. The isotropic chiral currents (9.2) are separately conserved and the coupling vanishes when  $\rho = 4 |\xi^\dagger \eta| = 0$ . This happens for  $\xi = 0$  or  $\eta = 0$ , (separated chiral components), or in the so called Majorana case which cannot be developed here [28],[29] :

$$\xi = \mathbf{f}(\mathbf{x}, t) \mathbf{s}_2 \eta^* \Rightarrow \xi = e^{i\theta(\mathbf{x}, t)} \mathbf{s}_2 \eta^* \quad (12.3)$$

Now, in (12.2)  $\xi$  and  $\eta$  are phase independent. In *the absence of external field*, the plane waves are :

$$\xi = \mathbf{a} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} ; \quad \eta = \mathbf{b} e^{i(\omega' t - \mathbf{k}' \cdot \mathbf{r})} \quad (12.4)$$

which gives the dispersion relation [7], [8], [21]:

$$\left(\frac{\omega^2}{c^2} - \mathbf{k}^2\right)\left(\frac{\omega'^2}{c^2} - \mathbf{k}'^2\right) - 2\left(\frac{\omega\omega'}{c^2} - \mathbf{k}\mathbf{k}'\right)\kappa^2(\rho) + \kappa^4(\rho) = 0 ; \kappa(\rho) = \frac{dF(\rho)}{d\rho} \quad (12.5)$$

If shall consider the case of an equation homogeneous in  $\xi$  and  $\eta$ :

$$F(\rho) = \kappa_0 \rho ; \kappa(\rho) = \kappa_0 = \text{Const} \quad (12.6)$$

Two kinds of waves (14.1) are particularly interesting :

1)  $\omega = \omega'$ ,  $\mathbf{k} = \mathbf{k}'$  : both monopoles have the same phase and the dispersion relation reduces to :

$$\frac{\omega^2}{c^2} = k^2 + \kappa_0^2 ; \left(k = \sqrt{k^2}\right) \quad (12.5)$$

This is the ordinary dispersion relation of a massive particle : a *bradyon*.

2)  $\omega = -\omega'$ ,  $\mathbf{k} = -\mathbf{k}'$  . The phases have opposite signs and the dispersion relation becomes :

$$\frac{\omega^2}{c^2} = k^2 - \kappa_0^2 \quad (12.6)$$

This is the dispersion relation of a supraluminal particle, a *tachyon*. The wave equation (13.2) seems to be the first one in which tachyons appear, although it was written for quite independent reasons. Actually, this non linear equation has many other possibilities that are more accurately described in the works quoted in the References, especially [7].

Nevertheless, let us finish by an important remark concerning **the nonlinear monopole in a coulombian electric field**. Chiral components of (12.2) cannot be separated as they were in the linear case (8.4). We must go back to the  $\Psi$  representation (6.23) that gives equivalently to (12.2) :

$$\gamma_\mu \left( \partial_\mu - \frac{g}{\hbar c} \gamma_3 \mathbf{B}_\mu \right) \Psi + \kappa(\rho) \frac{\Omega_1 - i \gamma_3 \Omega_2}{\sqrt{\Omega_1^2 + \Omega_2^2}} = 0 \quad (12.7)$$

In a coulombian electric field with a pseudo-potential (5.5), the angular operators (11.1) in the  $\Psi$  representation is :

$$\mathbf{J} = \hbar \left[ \mathbf{r} \times (-i \nabla + \gamma_4 \delta \mathbf{B}) + \gamma_4 \delta \frac{\mathbf{r}}{r} + \frac{1}{2} \mathbf{S} \right] ; \mathbf{S} = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix} ; \delta = \frac{eg}{\hbar c} ; \mathbf{B} = eB \quad (12.8)$$

This formula combines the preceding 2x2 operators (11.1).  $\mathbf{J}$  commutes with the *linear Hamiltonian* and is an integral of motion of (6.23). But we cannot hope a "commutation" with the nonlinear Hamiltonian of the equation (12.7).

In order to prove that  $\mathbf{J}$  is also an integral of the nonlinear system, *we must go back to the definition of an integral of motion* and directly verify that the *mean value* of the operator  $\mathbf{J}$  is a constant in virtue of the wave equations (12.7). And it is just which happens. If  $\psi$  is a solution of the eq. (12.7) one finds indeed :

$$\frac{\partial}{\partial t} \int \Psi^+ \mathbf{J} \Psi \, dx dy dz = 0 \quad (12.9)$$

So, the nonlinear equation (12.7) defines the same angular momentum as the linear equation (6.23). The difference will be only in the radial factor.

### 13. Chiral gauge and twisted space.

Let us take the particular case of (12.7) when  $B_\mu = O$ ,  $\kappa(\rho) = \lambda\rho$ ,  $\lambda = const$  :

$$\gamma_\mu \partial_\mu \Psi + \lambda (\Omega_1 - i \gamma_5 \Omega_2) \Psi = 0 \quad (13.1)$$

Under formally different but actually equivalent forms this equation was considered by several authors ([31 - [37]) among whom Weyl and Heisenberg. Despite the differences between (13.1) and (12.7) (particularly  $B_\mu = O$  : no external field), this case is interesting, especially in the paper of Rodichev [37] who considered  $\{x, y, z, t\}$  as a space with affine connection. Let us briefly recall :

1) The **affine connection** means that there are contravariant and covariant vectors  $T^i$  and  $T_i$ , and *covariant derivatives* which are tensors, while the *connection coefficients*  $\Gamma_{rk}^i$  are not. :

$$\nabla_\mu \Psi = \partial_\mu \Psi - \frac{i}{4} \Phi_{[\mu\nu\lambda]} \gamma_\nu \gamma_\lambda \Psi \quad (13.1)$$

No metric is supposed and if there is one, the  $g_{ij}$  are not (a priori) related to the  $\Gamma_{rk}^i$ .

2) Two tensors are defined<sup>5</sup> :

$$\text{- curvature : } -R_{qkl}^i = \frac{\partial \Gamma_{ql}^i}{\partial x^k} - \frac{\partial \Gamma_{qk}^i}{\partial x^l} + \Gamma_{pk}^i \Gamma_{ql}^p - \Gamma_{pl}^i \Gamma_{qk}^p \quad (13.2)$$

$$\text{- and torsion : } S_{[\mu\nu]}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \quad (13.3)$$

3) A **parallel transport** along a direction  $\xi^k = \frac{dx^k}{dt}$  is defined as :  $\nabla_{\xi^k} T = \xi^k \nabla_k T = 0$ . The transported vector depends on the curve along which it is transported. A **geodesic line** is enveloped by the parallel transport of its tangent, but except in a euclidian space, a *geodesic rectangle is not closed* because in an infinitesimal rectangle, there is a gap in 2 terms, the first (the greatest) is in  $dt^2$ , it depends on torsion, the second (the smaller) is in  $o(dt^3)$  and depends on curvature :

$$x_p^k - x_r^k = S_{ij}^k \left( \frac{dx^i}{dt} \right)_p \left( \frac{dx^j}{dt} \right)_p dt^2 + o(dt^3) \quad (13.4)$$

4) A **twisted space** is a space which has a torsion :  $S_{[\mu\nu]}^\lambda \neq 0$ . ( $\Gamma_{[\mu\nu]}^\lambda = S_{\mu\nu}^\lambda \neq 0$ ) In such a space a geodesic loop is an arc of helicoid the « thread » of which, at the infinitesimal limit, is of the *second order* – the order the area - while in a symmetric space an infinitesimal gap is of higher order. Approximately the same happens in a *spin fluid* : the ordinary angular momentum of a droplet is of higher order than the spin of the droplet [38], [39], [40].

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<sup>5</sup> If  $R_{qkl}^i = S_{[\mu\nu]}^\lambda = 0$  the space is euclidian.

Now Rodichev considers a **flat twisted space** : with straight geodesics ( $\Gamma_{(\mu\nu)}^\lambda = 0$ ), and with torsion ( $\Gamma_{[\mu\nu]}^\lambda = S_{\mu\nu}^\lambda \neq 0$ ) and he chooses the following connection and covariant spinor derivative :

$$\Gamma_{\lambda[\mu\nu]} = S_{\lambda\mu\nu} = \Phi_{[\lambda\mu\nu]} ; \nabla_\mu \Psi = \partial_\mu \Psi - \frac{i}{4} \Phi_{[\mu\nu\lambda]} \gamma_\nu \gamma_\lambda \Psi \quad (13.5)$$

and the Lagrangian density :

$$L = \frac{1}{2} \left\{ \bar{\Psi} \gamma_\mu \nabla_\mu \Psi - (\nabla_\mu \bar{\Psi}) \gamma_\mu \Psi \right\} \quad (13.5)$$

Let us make a short remark about that, translating the formula (13.5) in our language, which gives:

$$L = \frac{1}{2} \left\{ \bar{\Psi} \gamma_\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \gamma_\mu \Psi - \frac{i}{2} \Phi_{[\mu\nu\lambda]} \bar{\Psi} \gamma_\nu \gamma_\lambda \Psi \right\} \quad (13.6)$$

Introducing the axial dual vector:  $\Phi_\mu = \frac{i}{3!} \varepsilon_{[\mu\nu\lambda\sigma]} \Phi_{[\nu\lambda\sigma]}$ , the lagrangian becomes :

$$L = \frac{1}{2} \left\{ \bar{\Psi} \gamma_\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \gamma_\mu \Psi - \frac{1}{2} \Phi_\mu \bar{\Psi} \gamma_\mu \gamma_5 \Psi \right\} \quad (13.7)$$

from which we derive the equation :

$$\gamma_\mu \left( \partial_\mu - \frac{1}{2} \Phi_\mu \gamma_5 \right) \Psi = 0 \quad (13.8)$$

With  $\Phi_\mu = \frac{2g}{\hbar c} B_\mu$ , this is *the equation (6.23)*. And here,  $\Phi_\mu$  is not introduced as an external field, but as an intrinsic geometrical property, so we can say that **a monopole plunged in an electromagnetic field induces a torsion in the surrounding space.**

Now we go back to the reasoning of Rodichev who ignored the monopole and didn't aimed at the linear equation (13.8), but at a nonlinear equation. He considered the following Einstein-like action integral :

$$S = \int (L - bR) d^4x \quad (13.9)$$

$L$  is given by (13.5) without external field,  $b = \text{Const}$ ,  $R = \text{total curvature}$  and, in virtue of (13.3) :

$$R = \Phi_{[\lambda\mu\nu]} \Phi_{[\lambda\mu\nu]} \text{ or, in terms of dual vector : } R = -6\Phi_\mu \Phi_\mu \quad (13.10)$$

Hence, (13.9) becomes :

$$S = \int \left\{ \frac{1}{2} \left[ \bar{\Psi} \gamma_\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \gamma_\mu \Psi - 2\Phi_\mu \bar{\Psi} \gamma_\mu \gamma_5 \Psi \right] + 36b \Phi_\mu \Phi_\mu \right\} d^4x \quad (13.11)$$

Now, if we vary  $S$ , not with respect to  $\Psi$  with respect to  $\Phi$ , we find :

$$\Phi_{\mu} = \frac{1}{18b} \bar{\Psi} \gamma_{\mu} \gamma_5 \Psi \quad (13.12)$$

and variation of  $\Psi$  gives:

$$\gamma_{\lambda} \partial_{\lambda} - \frac{1}{9b} (\bar{\Psi} \gamma_{\mu} \gamma_5 \Psi) \gamma_{\mu} \gamma_5 \Psi = 0 \quad (13.13)$$

So doing, we come back once more to the monopole, and now in the nonlinear case because, up to a constant factor, (13.15) is identic to (13.1) which is a particular case of the nonlinear equation (12.7). The identity between (13.13) and (13.1) is due to the identities (6.13), in virtue of which and of (13.14) :

$$R = \frac{-1}{54b} (\Omega_2^2 + \Omega_2^2) \quad (13.15)$$

Which means that the chiral invariant  $(\Omega_2^2 + \Omega_2^2)$ , apart from a constant factor, is the curvature of the twisted space created by the self action of the monopole.

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