Formulaire de MECA 2100 [45-45] et [30-30]
Mécanique des Solides Déformables

Version de juin 2004

Ce formulaire ne vous appartient pas.
Prière de ne rien écrire dessus et de ne pas l'emporter.

1 Equations fondamentales de l'élasticité linéaire et isotrope

1.1 Tenseur contraintes de Cauchy $\boldsymbol{\sigma}$

1.1.1 Vecteurs contraintes ("forces de contact")

$$\boldsymbol{t} = \boldsymbol{\sigma}^T.\boldsymbol{n}, \qquad t_i = \sigma_{ji} n_j = \sigma n_i + \tau_i, \qquad \sigma = \boldsymbol{n}^T.\boldsymbol{\sigma}^T.\boldsymbol{n} = n_i \sigma_{ji} n_j$$

1.1.2 Contraintes principales, invariants, (tri-)cercle de Mohr

$$\det(\boldsymbol{\sigma} - \lambda \boldsymbol{I}) = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0 \quad \Longrightarrow \quad \sigma_1, \sigma_2, \sigma_3;$$

$$\begin{aligned}
I_1 &= \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \frac{1}{2}\left(I_1^2 - \sigma_{ij}\sigma_{ji}\right) = \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \\
&= \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2 \\
I_3 &= \det \boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} = \sigma_1\sigma_2\sigma_3
\end{aligned}$$

En 2D :

$$\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

1.1.3 Equations d'équilibre

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}, \qquad \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0; \qquad\qquad \boldsymbol{\sigma}^T = \boldsymbol{\sigma}, \qquad \sigma_{ji} = \sigma_{ij}$$

1.2 Déformations inifinitésimales

1.2.1 Relation déplacements-déformations

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{grad}\,\boldsymbol{u} + (\mathbf{grad}\,\boldsymbol{u})^T\right), \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

1.2.2 Equations de compatibilité de Saint-Venant

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_l \partial x_l} - \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} = 0$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} + \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_1} - \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_3} - \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_2} = 0$$
$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} + \frac{\partial^2 \varepsilon_{31}}{\partial x_2 \partial x_2} - \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_1} - \frac{\partial^2 \varepsilon_{21}}{\partial x_2 \partial x_3} = 0$$
$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} + \frac{\partial^2 \varepsilon_{12}}{\partial x_3 \partial x_3} - \frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_2} - \frac{\partial^2 \varepsilon_{32}}{\partial x_3 \partial x_1} = 0$$

$$2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_2} - \frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_1} = 0$$
$$2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} - \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_3} - \frac{\partial^2 \varepsilon_{33}}{\partial x_2 \partial x_2} = 0$$
$$2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} - \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_1} - \frac{\partial^2 \varepsilon_{11}}{\partial x_3 \partial x_3} = 0$$

1.3 Elasticité linéaire isotrope

1.3.1 Loi de Hooke

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu(\varepsilon_{22}+\varepsilon_{33})\right], \ \ldots \qquad \sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12}, \ \ldots$$

$$\sigma_{kk} = 3K\varepsilon_{kk}, \qquad s_{ij} = 2\mu e_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})\right], \ \ldots \qquad \varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}, \ \ldots$$

$$\varepsilon_{kk} = \frac{1}{3K}\sigma_{kk}, \qquad e_{ij} = \frac{1}{2\mu}s_{ij}$$

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}, \qquad e_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{kk}}{3}\delta_{ij}.$$

1.3.2 Coefficients d'élasticité

Module de Young E et coefficient de Poisson ν :

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{9KG}{3K+G}, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{3K-2G}{6K+2G}$$

Constantes de Lamé λ et μ :

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)} = K - \frac{2}{3}G, \qquad 2\mu = \frac{E}{1+\nu} = 2G$$

Module de compressibilité K et module de Coulomb ou de cisaillement G :

$$3K = \frac{E}{1-2\nu} = 3\lambda + 2\mu, \qquad\qquad 2G = \frac{E}{1+\nu} = 2\mu$$

1.4 Critères de plasticité

1.4.1 Contrainte équivalente et critère de Von Mises

$$\begin{aligned}\sigma_{\mathrm{VM}} &= \sqrt{\frac{3}{2} s_{ij} s_{ij}} \\ &= \left\{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right]\right\}^{1/2}\end{aligned}$$

$$\sigma_{\mathrm{VM}} \leq \sigma_Y$$

1.4.2 Energie de distorsion. Contrainte octaédrale

$$W_{\mathrm{dist}} = \mu e_{ij} e_{ij} = \frac{1}{4\mu} s_{ij} s_{ij} = \frac{\sigma_{\mathrm{VM}}^2}{6\mu}; \qquad\qquad \tau_{\mathrm{oct}} = \frac{\sqrt{2}}{3}\sigma_{\mathrm{VM}}$$

1.4.3 Contrainte tangentielle maximale. Critère de Tresca

$$\tau_{\max} = \max\left\{\left|\frac{\sigma_1-\sigma_2}{2}\right|, \left|\frac{\sigma_2-\sigma_3}{2}\right|, \left|\frac{\sigma_3-\sigma_1}{2}\right|\right\}; \qquad\qquad \tau_{\max} \leq \frac{\sigma_Y}{2}$$

1.5 Problèmes d'élasticité linéaire

1.5.1 Navier : équilibre en termes de déplacements

$$(\lambda+\mu)\,\mathbf{grad}\,(\mathrm{div}\,\boldsymbol{u}) + \mu\Delta\boldsymbol{u} + \boldsymbol{f} = \mathbf{0}, \qquad (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i = 0$$

1.5.2 Beltrami-Mitchell : compatibilité en termes de contraintes

$$\Delta\boldsymbol{\sigma} + \frac{1}{1+\nu}\,\mathbf{grad}\,\mathbf{grad}\,\mathrm{tr}\,(\boldsymbol{\sigma}) = -\frac{\nu}{1-\nu}\,\mathrm{div}\,\boldsymbol{f}\boldsymbol{I} - \left(\mathbf{grad}\,\boldsymbol{f} + (\mathbf{grad}\,\boldsymbol{f})^T\right)$$

$$\frac{\partial^2\sigma_{ij}}{\partial x_k \partial x_k} + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial x_i \partial x_j} = -\frac{\nu}{1-\nu}\frac{\partial f_k}{\partial x_k}\delta_{ij} - \left(\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right)$$

$$\mathbf{grad}\,\boldsymbol{f} = \mathbf{0}, \qquad \frac{\partial f_i}{\partial x_j} = 0 \quad\Longrightarrow\quad \Delta\Delta\boldsymbol{\sigma} = 0, \qquad \frac{\partial^4\sigma_{ij}}{\partial x_k \partial x_k \partial x_l \partial x_l} = 0$$

2 Formulations variationnelles et introduction aux méthodes numériques

2.1 Formulation locale ("forte")

Trouver $\boldsymbol{u}, \boldsymbol{\sigma}$ tels que :

$$\begin{cases} \dfrac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 & \text{dans } \Omega \\ \sigma_{ij} n_j = \bar{t}_i & \text{sur } \Gamma_F \\ u_i = \bar{u}_i & \text{sur } \Gamma_U \end{cases}$$

et

$$\sigma_{ij} = C_{ijkl} u_{(k,l)}, \qquad u_{(k,l)} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = \varepsilon_{kl}$$

2.2 Formulation variationnelle ("faible") en déplacements

Trouver $\boldsymbol{u}$ cinématiquement admissible tel que :

$$\underbrace{\int_\Omega \sigma_{ij} v_{(i,j)} d\Omega}_{\delta W} = \underbrace{\int_\Omega f_i v_i d\Omega + \int_{\Gamma_F} \bar{t}_i v_i d\Gamma_F + \int_{\Gamma_U} \bar{u}_i \sigma_{ij} n_j d\Gamma_U}_{\delta \mathcal{T}}$$

pour tout $\boldsymbol{v}$ cinématiquement admissible, avec :

$$\sigma_{ij} = C_{ijkl} u_{(k,l)}$$

2.3 Théorème de l'énergie potentielle en déplacements

Trouver $\boldsymbol{u}$ cinématiquement admissible tel que :

$$I(\boldsymbol{u}) = \underbrace{\frac{1}{2}\int_\Omega u_{(i,j)} C_{ijkl} u_{(k,l)} d\Omega}_{W(\boldsymbol{u})} \underbrace{- \int_\Omega f_i u_i d\Omega - \int_{\Gamma_F} \bar{t}_i u_i d\Gamma_F}_{-\mathcal{T}(\boldsymbol{u})} = \min$$

2.4 Méthode approchée de Ritz

Pour un ensemble de n champs $\boldsymbol{u}^{(i)}$ choisis, cinématiquement admissibles et linéairement indépendants :

$$\boldsymbol{u} \simeq \tilde{\boldsymbol{u}} = \sum_{i=1}^{n} a_i \boldsymbol{u}^{(i)} \quad \text{tel que} \quad I(\tilde{\boldsymbol{u}}) = \min \quad \Leftrightarrow \quad \frac{\partial I(\tilde{\boldsymbol{u}})}{\partial a_i} = 0 \quad i = 1, \ldots, n$$

2.5 Formulation variationnelle ("faible") en contraintes

Trouver $\boldsymbol{\sigma}$ statiquement admissible tel que :

$$\int_\Omega \tau_{ij} u_{(i,j)} d\Omega = \int_\Omega -\frac{\partial \tau_{ij}}{\partial x_j} u_i d\Omega + \int_{\Gamma_F} u_i \tau_{ij} n_j d\Gamma_F + \int_{\Gamma_U} \bar{u}_i \tau_{ij} n_j d\Gamma_U$$

pour tout $\boldsymbol{\tau}$ statiquement admissible, avec :

$$u_{(i,j)} = S_{ijkl} \sigma_{kl}, \qquad \boldsymbol{S} = \boldsymbol{C}^{-1}$$

2.6 Théorème de l'énergie complémentaire en contraintes

Trouver $\boldsymbol{\sigma}$ statiquement admissible tel que :

$$J(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \sigma_{ij} S_{ijkl} \sigma_{kl} d\Omega + \int_{\Gamma_U} \bar{u}_i \sigma_{ij} n_j d\Gamma_U = \min$$

2.7 Encadrement

La solution $\boldsymbol{u}$ (cinématiquement admissible) et $\boldsymbol{\sigma}$ (statiquement admissible) est telle que :

$$-J(\boldsymbol{\tau}) \leq -J(\boldsymbol{\sigma}) = I(\boldsymbol{u}) \leq I(\boldsymbol{v})$$

$\forall\, \boldsymbol{\tau}$ statiquement admissible et $\forall\, \boldsymbol{v}$ cinématiquement admissible.

$$\underbrace{\int_\Omega \sigma_{ij} u_{(i,j)} d\Omega}_{2W} = \underbrace{\int_\Omega f_i u_i d\Omega + \int_{\Gamma_F} \bar{t}_i u_i d\Gamma_F}_{\mathcal{T}_F(\boldsymbol{u})} + \underbrace{\int_{\Gamma_U} \bar{u}_i \sigma_{ij} n_j d\Gamma_U}_{\mathcal{T}_R(\boldsymbol{\sigma})}$$

3 Théorie des poutres ou résistance des matériaux

3.1 Efforts intérieurs. Equations d'équilibre

$$N(x) = \int_S \sigma_{xx}\, dS, \qquad Q(x) = \int_S \sigma_{xy}\, dS, \qquad M(x) = \int_S y\sigma_{xx}\, dS.$$

$$\frac{dN}{dx} + f_x(x) = 0, \qquad \frac{dQ}{dx} + f_y(x) = 0, \qquad \frac{dM}{dx} = Q(x).$$

$$N^+ - N^- + F_x = 0, \qquad Q^+ - Q^- + F_y = 0, \qquad M^+ - M^- - C_z = 0.$$

$$\sigma_{xx}^{(N)} = \frac{N(x)}{A}, \qquad \sigma_{xy}^{(Q)} = \frac{Q(x)}{I_z b(y)} \int_{S^*(y)} y^* dS^*, \qquad \sigma_{xx}^{(M)} = \frac{M(x)y}{I_z}.$$

$$A = \int_S dS, \qquad (y = 0 \text{ au C.G.}) \qquad I_z = \int_S y^2 dS.$$

$$\frac{d^2M}{dx^2} = -f_y(x); \qquad M(x) = -EI_z \frac{d^2v}{dx^2}; \qquad \frac{d^2}{dx^2}\left(EI_z \frac{d^2v}{dx^2}\right) = f_y(x)$$

3.2 Théorème de Maxwell-Betti

$$\sum F_y^{(1)} v^{(2)} + \sum C^{(1)}\theta^{(2)} + \int_\Gamma f_y^{(1)} v^{(2)} d\Gamma + \cdots$$
$$= \sum F_y^{(2)} v^{(1)} + \sum C^{(2)}\theta^{(1)} + \int_\Gamma f_y^{(2)} v^{(1)} d\Gamma + \cdots$$

3.3 Théorème de Castigliano

$$W^e = \int_\Gamma \left\{ \frac{M^2}{2EI} + \frac{N^2}{2EA} + \frac{Q^2}{2GA_1} \right\} d\Gamma + \underbrace{\sum \frac{F^2}{2k_F}}_{\substack{\text{si ressorts}\\ \text{linéaires}}} + \underbrace{\sum \frac{C^2}{2k_C}}_{\substack{\text{si ressorts}\\ \text{spiraux}}} + \cdots$$

$$d_R = \frac{\partial W^e}{\partial R}, \qquad d_R \text{ associé à } R \text{ (mêmes direction et sens)}$$

4 Torsion des poutres

4.1 Angle de rotation unitaire α

$$u(x,y,z) = \alpha\phi(y,z), \qquad v(x,y,z) = -\alpha xz, \qquad w(x,y,z) = \alpha yx$$

4.2 Fonction de contraintes de Prandtl $\Psi(y,z)$

$$\Delta\Psi = -2 \quad \text{dans } \mathcal{S};$$

Coordonnées cartésiennes

$$\sigma_{xz} = -\mu\alpha\frac{\partial\Psi}{\partial y}, \qquad \sigma_{xy} = +\mu\alpha\frac{\partial\Psi}{\partial z},$$

$$\frac{\partial\phi}{\partial y} = +\frac{\partial\Psi}{\partial z} + z - z_G,$$

$$\frac{\partial\phi}{\partial z} = -\frac{\partial\Psi}{\partial y} - y + y_G,$$

$$\vec{t} = \sigma_{yx}\boldsymbol{e}_y + \sigma_{zx}\boldsymbol{e}_z;$$

Coordonnées cylindriques

$$\sigma_{rx} = +\mu\alpha\frac{1}{r}\frac{\partial\Psi}{\partial\theta}, \qquad \sigma_{\theta x} = -\mu\alpha\frac{\partial\Psi}{\partial r},$$

$$\frac{\partial\phi}{\partial r} = +\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + y_G\sin\theta - z_G\cos\theta,$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r} - r + y_G\cos\theta + z_G\sin\theta,$$

$$\vec{t} = \sigma_{rx}\boldsymbol{e}_r + \sigma_{\theta x}\boldsymbol{e}_\theta.$$

4.2.1 Section simplement connexe

$$\Psi(y,z) = 0 \quad \text{sur } \mathcal{C}; \qquad \frac{M}{\alpha} = 2\mu\int_{\mathcal{S}}\Psi\,d\mathcal{S}$$

4.2.2 Section multiplement connexe

$$\Psi(y,z) = 0 \quad \text{sur } \mathcal{C}, \text{ et} \quad \Psi(y,z) = k_i \quad \text{sur } \mathcal{C}_i, \text{ avec} \quad \oint_{\mathcal{C}_i}\frac{\partial\Psi}{\partial n}d\mathcal{C}_i = 2A_i$$

$$\frac{M}{\alpha} = 2\mu\int_{\mathcal{S}}\Psi\,d\mathcal{S} + 2\mu\sum k_i A_i$$

4.3 Energie potentielle

$$\mathcal{I} = \frac{1}{2}\mu l\alpha^2\int_S\left[(\frac{\partial\Psi}{\partial y})^2 + (\frac{\partial\Psi}{\partial z})^2\right]dy\,dz - M\alpha l$$

$$\begin{aligned} Ml\alpha &= 2\mu l\alpha^2\int_S\Psi(y,z)dy\,dz && \text{(simplement connexe)} \\ &= 2\mu l\alpha^2\left[\int_S\Psi(y,z)dy\,dz + \sum_{i=1}^{n}k_iA_i\right] && \text{(multiplement connexe)} \end{aligned}$$

5 Flexion des plaques minces de Kirchhoff-Love

5.1 Efforts intérieurs. Equations d'équilibre

$$Q_1 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xz}\, dz, \qquad Q_2 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yz}\, dz$$

$$M_{11} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_{xx}\, dz, \qquad M_{22} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_{yy}\, dz, \qquad M_{12} = M_{21} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_{xy}\, dz$$

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + f_3 = 0, \qquad \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = Q_1, \qquad \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = Q_2$$

5.2 Déformations. Courbures de la surface médiane

$$\tilde{\varepsilon} = \begin{bmatrix} \tilde{\varepsilon}_{xx} & \tilde{\varepsilon}_{xy} \\ \tilde{\varepsilon}_{xy} & \tilde{\varepsilon}_{yy} \end{bmatrix} = z\boldsymbol{\kappa}, \qquad \boldsymbol{\kappa} = -\begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} & \dfrac{\partial^2 w}{\partial x \partial y} \\ \dfrac{\partial^2 w}{\partial x \partial y} & \dfrac{\partial^2 w}{\partial y^2} \end{bmatrix} = -\boldsymbol{\nabla}\boldsymbol{\nabla}^T w$$

5.3 Loi de Hooke. Rigidité flexionnelle

$$\boldsymbol{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \mathcal{D}\left[(1-\nu)\,\boldsymbol{\kappa} + \nu\,\mathrm{tr}\,(\boldsymbol{\kappa})\,\boldsymbol{I}\right]; \qquad \mathcal{D} = \frac{Eh^3}{12\,(1-\nu^2)}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} = \frac{12}{h^3} z\boldsymbol{M}$$

5.4 Equation fondamentale. Conditions aux limites

$$\Delta\Delta w(x,y) = \frac{f_z(x,y)}{\mathcal{D}}$$

- Appui simple: $w = 0$ et $M_{nn} = 0$;

bord droit en $x = a$: $w = 0$ et $\dfrac{\partial^2 w}{\partial x^2} = 0.$

- Encastrement : $w = 0$ et $\dfrac{\partial w}{\partial n} = 0.$

- Bord libre : $Q_x + \dfrac{\partial M_{ns}}{\partial s} = 0$ et $M_{nn} = 0$;

bord droit en $x = a$: $\dfrac{\partial^3 w}{\partial x^3} + (2-\nu)\dfrac{\partial^3 w}{\partial x \partial y^2} = 0$ et $\dfrac{\partial^2 w}{\partial x^2} + \nu\dfrac{\partial^2 w}{\partial y^2} = 0.$

5.5 Solution de Lévy pour les plaques rectangulaires sur 2 appuis simples opposés en $x = 0, a$

$$f_z = p\,(x,y)\,; \qquad w\,(x,y) = w_1\,(x,y) + w_2\,(x,y)\,; \qquad \Delta\Delta w_1 = \frac{p\,(x,y)}{\mathcal{D}}, \qquad \Delta\Delta w_2 = 0.$$

5.5.1 Solution particulière w_1 respectant les C.L. en $x = 0, a$

$$w_1(x,y) = \sum_{n=1}^{\infty} k_n(y) \sin \frac{n\pi x}{a}$$

avec $k_n(y)$ solution particulière de :

$$\frac{d^4 k_n}{dy^4} - 2\left(\frac{n\pi}{a}\right)^2 \frac{d^2 k_n}{dy^2} + \left(\frac{n\pi}{a}\right)^4 k_n = \frac{p_n(y)}{\mathcal{D}}$$

avec :

$$p(x,y) = \sum_{n=1}^{\infty} p_n(y) \sin \frac{n\pi x}{a} \quad \Rightarrow \quad p_n(y) = \frac{2}{a}\int_0^a p(x,y) \sin \frac{n\pi x}{a}\, dx$$

Cas particulier $p = p(x)$:

$$k_n = \left(\frac{a}{n\pi}\right)^4 \frac{p_n}{\mathcal{D}}; \qquad\qquad w_1 = w_1(x) = \sum_{n=1}^{\infty} \left(\frac{a}{n\pi}\right)^4 \frac{p_n}{\mathcal{D}} \sin \frac{n\pi x}{a}$$

5.5.2 Solution homogène w_2 respectant les C.L. en $x = 0, a$

$$w_2(x,y) = \sum_{n=1}^{\infty} Y_n(y) \sin \frac{n\pi x}{a}$$

avec $Y_n(y)$ solution de :

$$\frac{d^4 Y_n}{dy^4} - 2\left(\frac{n\pi}{a}\right)^2 \frac{d^2 Y_n}{dy^2} + \left(\frac{n\pi}{a}\right)^4 Y_n = 0$$

$$\begin{aligned} Y_n(y) &= \left(a_n + b_n \frac{n\pi y}{a}\right) \exp\left(-\frac{n\pi y}{a}\right) + \left(c_n + d_n \frac{n\pi y}{a}\right) \exp\left(+\frac{n\pi y}{a}\right) \\ &= A_n \cosh \frac{n\pi y}{a} + B_n \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + C_n \frac{n\pi y}{a} \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a} \end{aligned}$$

5.5.3 Conditions aux limites $y = \pm b/2$

A respecter par $w(x,y) = w_1(x,y) + w_2(x,y)$ pour $0 \leq x \leq a$
$\Rightarrow$ déterminent a_n, b_n, c_n et d_n ou A_n, B_n, C_n et D_n.

5.6 Energie potentielle

$$\begin{aligned} \mathcal{I}(w) &= \mathcal{W}(w) - \int_S pw\, dx\, dy \text{ (charge surfacique)} \\ &= \mathcal{W}(w) - Pw(x_P, y_P) \text{ (charge concentrée)} \end{aligned}$$

$$\mathcal{W}(w) = \int_S \frac{\mathcal{D}}{2} \left\{ (\Delta w)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\, dy$$

6 Flexion de plaques minces en coordonnées cylindriques

6.1 Efforts intérieurs. Equations d'équilibre

$$Q_r = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{rz}\, dz, \qquad Q_\theta = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\theta z}\, dz \qquad \frac{\partial Q_r}{\partial r} + \frac{1}{r}\left(\frac{\partial Q_\theta}{\partial \theta} + Q_r\right) + f_z = 0$$

$$M_{rr} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_{rr}\, dz, \qquad M_{\theta\theta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_{\theta\theta}\, dz, \qquad M_{r\theta} = M_{\theta r} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_{r\theta}\, dz$$

$$\frac{\partial M_{rr}}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial \theta} + \frac{M_{rr} - M_{\theta\theta}}{r} = Q_r, \qquad \frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{2M_{r\theta}}{r} = Q_\theta$$

6.2 Courbures de la surface médiane. Loi de Hooke

$$\kappa_{rr} = -\frac{\partial^2 w}{\partial r^2}, \qquad \kappa_{\theta\theta} = -\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial w}{\partial r}\right), \qquad \kappa_{r\theta} = \kappa_{\theta r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$\boldsymbol{M}_{(r,\theta)} = \mathcal{D}\left[(1-\nu)\,\boldsymbol{\kappa}_{(r,\theta)} + \nu\left(\kappa_{rr} + \kappa_{\theta\theta}\right)\boldsymbol{I}\right], \qquad \boldsymbol{\sigma}_{(r,\theta)} = \frac{12}{h^3} z \boldsymbol{M}_{(r,\theta)}$$

6.3 Equation fondamentale

$$\Delta\Delta w(r,\theta) = \frac{f_z(r,\theta)}{\mathcal{D}}, \qquad \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

6.4 Problèmes axisymétriques

6.4.1 Cas général

$$w = w(r), \qquad M_{r\theta} = M_{\theta r} = 0, \qquad Q_\theta = 0.$$

$$\frac{dQ_r}{dr} + \frac{Q_r}{r} + f_z = 0, \qquad \frac{dM_{rr}}{dr} + \frac{M_{rr} - M_{\theta\theta}}{r} = Q_r.$$

$$M_{rr} = -\mathcal{D}\left(\frac{d^2 w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right), \qquad M_{\theta\theta} = -\mathcal{D}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2 w}{dr^2}\right), \qquad Q_r = -\mathcal{D}\frac{d}{dr}\left(\Delta w\right)$$

$$\sigma_{rr} = \frac{12}{h^3} z M_{rr}, \qquad \sigma_{\theta\theta} = \frac{12}{h^3} z M_{\theta\theta}, \qquad \sigma_{r\theta} = \sigma_{\theta r} = 0.$$

$$\Delta\Delta w(r) = \frac{f_z(r)}{\mathcal{D}}, \qquad \Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

6.4.2 Charge uniforme $f_z = p$ sur une partie de la plaque :

$$w(r) = \frac{pr^4}{64\mathcal{D}} + Ar^2 + B\ln r + C + Dr^2\ln r; \qquad Q_r(r) = -\frac{pr}{2} - \mathcal{D}\frac{4D}{r}$$

$$M_{rr}(r) = -(3+\nu)\frac{pr^2}{16} - 2(1+\nu)\mathcal{D}A + (1-\nu)\mathcal{D}\frac{B}{r^2} - [2(1+\nu)\ln r + 3 + \nu]\mathcal{D}D$$

$$M_{\theta\theta}(r) = -(1+3\nu)\frac{pr^2}{16} - 2(1+\nu)\mathcal{D}A - (1-\nu)\mathcal{D}\frac{B}{r^2} - [2(1+\nu)\ln r + 1 + 3\nu]\mathcal{D}D$$

7 Problèmes plans en coordonnées cartésiennes

7.1 Equations générales

Déformations planes	Contraintes planes

7.1.1 Hypothèses

$$f_x = f_x(x,y), \qquad f_y = f_y(x,y), \qquad f_z = 0; \qquad u_x = u_x(x,y), \qquad u_y = u_y(x,y),$$

Déformations planes	Contraintes planes
$u_z = 0, \qquad \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0.$	$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0.$

7.1.2 Equations d'équilibre

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + f_x = 0, \qquad \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + f_y = 0.$$

7.1.3 Modèle de comportement

Déformations planes	Contraintes planes
$\varepsilon_{xx} = \frac{1+\nu}{E}[\sigma_{xx} - \nu(\sigma_{xx}+\sigma_{yy})]$	$\varepsilon_{xx} = \frac{1+\nu}{E}\left[\sigma_{xx} - \frac{\nu}{1+\nu}(\sigma_{xx}+\sigma_{yy})\right]$
$\varepsilon_{yy} = \frac{1+\nu}{E}[\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{yy})]$	$\varepsilon_{yy} = \frac{1+\nu}{E}\left[\sigma_{yy} - \frac{\nu}{1+\nu}(\sigma_{xx}+\sigma_{yy})\right]$
$\varepsilon_{zz} = 0 \Longrightarrow \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$	$\sigma_{zz} = 0 \Longrightarrow \varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy})$
$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$	$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$
$\varepsilon_{xz} = 0 \Longrightarrow \sigma_{xz} = 0$	$\sigma_{xz} = 0 \Longrightarrow \varepsilon_{xz} = 0$
$\varepsilon_{yz} = 0 \Longrightarrow \sigma_{yz} = 0$	$\sigma_{yz} = 0 \Longrightarrow \varepsilon_{yz} = 0$

7.1.4 Equation de compatibilité

Déformations planes	Contraintes planes
$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$	$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy}) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$

7.1.5 Substitutions (à effectuer en parallèle)

$$\left.\begin{array}{c}\text{Solution en}\\ \text{déformations}\\ \text{planes}\end{array}\right\} \begin{array}{ccccccc} \Longrightarrow & E \rightarrow \dfrac{1+2\nu}{(1+\nu)^2}E & ; & \nu \rightarrow \dfrac{\nu}{1+\nu} & \Longrightarrow \\ \Longleftarrow & \dfrac{1}{1-\nu^2}E \leftarrow E & ; & \dfrac{\nu}{1-\nu} \leftarrow \nu & \Longleftarrow \end{array} \left\{\begin{array}{c}\text{Solution en}\\ \text{contraintes}\\ \text{planes}\end{array}\right.$$

7.1.6 Relations déformations–déplacements

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

7.2 Fonction de contraintes d'Airy $\phi(x, y)$

7.2.1 Hypothèse supplémentaire

$$f_x = f_y = 0$$

7.2.2 Equations d'équilibre

$$\sigma_{xx} = +\frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_{yy} = +\frac{\partial^2 \phi}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

7.2.3 Equation de compatibilité

$$\Delta\Delta\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0.$$

7.2.4 C.L. statiques

$$\begin{array}{l} \sigma_{xx} n_x + \sigma_{xy} n_y = \bar{t}_x, \\ \sigma_{xy} n_x + \sigma_{yy} n_y = \bar{t}_y \end{array} \qquad \text{sur } \partial\Omega_F$$

7.2.5 C.L. cinématiques

$$u_x = \bar{u}_x, \qquad u_y = \bar{u}_y \qquad \text{sur } \partial\Omega_U$$

7.2.6 Fonctions polynômiales

$$\phi_2 = \frac{1}{2}a_2x^2 + b_2xy + \frac{1}{2}c_2y^2; \qquad \sigma_{xx} = c_2, \qquad \sigma_{yy} = a_2, \qquad \sigma_{xy} = -b_2.$$

$$\phi_3 = \frac{1}{6}a_3x^3 + \frac{1}{2}b_3x^2y + \frac{1}{2}c_3xy^2 + \frac{1}{6}d_3y^3$$
$$\sigma_{xx} = c_3x + d_3y, \qquad \sigma_{yy} = a_3x + b_3y, \qquad \sigma_{xy} = -b_3x - c_3y.$$

$$\phi_4 = \frac{1}{12}a_4x^4 + \frac{1}{6}b_4x^3y + \frac{1}{2}c_4x^2y^2 + \frac{1}{6}d_4xy^3 + \frac{1}{12}e_4y^4; \qquad a_4 + 2c_4 + e_4 = 0$$
$$\sigma_{xx} = -\frac{a_4 + e_4}{2}x^2 + d_4xy + e_4y^2 \qquad \sigma_{yy} = +a_4x^2 + b_4xy - \frac{a_4 + e_4}{2}y^2$$
$$\sigma_{xy} = -\frac{1}{2}b_4x^2 + (a_4 + e_4)\,xy - \frac{1}{2}d_4y^2$$

$$\phi_5 = \frac{1}{20}a_5x^5 + \frac{1}{12}b_5x^4y + \frac{1}{6}c_5x^3y^2 + \frac{1}{6}d_5x^2y^3 + \frac{1}{12}e_5xy^4 + \frac{1}{20}f_5y^5$$

$$3a_5 + 2c_5 + e_5 = 0, \qquad b_5 + 2d_5 + 3f_5 = 0.$$

$$\sigma_{xx} = +\frac{1}{3}c_5x^3 + d_5x^2y - (3a_5 + 2c_5)\,xy^2 + f_5y^3$$

$$\sigma_{yy} = +a_5x^3 - (2d_5 + 3f_5)\,x^2y + c_5xy^2 + \frac{1}{3}d_5y^3$$

$$\sigma_{xy} = +\left(\frac{2}{3}d_5 + f_5\right)x^3 - c_5x^2y - d_5xy^2 + \left(a_5 + \frac{2}{3}c_5\right)y^3$$

7.2.7 Séries de Fourier

$$-\frac{\ell}{2} \le x \le +\frac{\ell}{2}, \qquad -\frac{c}{2} \le y \le +\frac{c}{2}.$$

$$\begin{cases} q^+(x) = \frac{1}{2}q_0^+ + \sum_{n=1}^{\infty} q_n^+ \cos\frac{2n\pi x}{\ell} + \sum_{n=1}^{\infty} Q_n^+ \sin\frac{2n\pi x}{\ell} & \text{en } y = +\frac{c}{2} \qquad (-\boldsymbol{e}_y q^+) \\ q^-(x) = \frac{1}{2}q_0^- + \sum_{n=1}^{\infty} q_n^- \cos\frac{2n\pi x}{\ell} + \sum_{n=1}^{\infty} Q_n^- \sin\frac{2n\pi x}{\ell} & \text{en } y = -\frac{c}{2} \qquad (+\boldsymbol{e}_y q^-) \end{cases}$$

$$\phi(x,y) = \phi_0(x,y) + \sum_{n=1}^{\infty} f_n(y)\cos\frac{2n\pi x}{\ell} + \sum_{n=1}^{\infty} F_n(y)\sin\frac{2n\pi x}{\ell}$$

$$\Delta\Delta\phi_0 = 0$$

$$f_n(y) = \alpha_n \cosh my + \beta_n my \sinh my + \gamma_n \sinh my + \delta_n my \cosh my, \qquad m = \frac{2n\pi}{\ell};$$

$$(\sigma_{xx})_n = +m^2 \cos mx\,[\alpha_n \cosh my + \beta_n(2\cosh my + my\sinh my) + \gamma_n \sinh my + \delta_n(2\sinh my + my\cosh my)]$$

$$(\sigma_{yy})_n = -m^2 \cos mx\,[\alpha_n \cosh my + \beta_n my\sinh my + \gamma_n \sinh my + \delta_n my\cosh my]$$

$$(\sigma_{xy})_n = +m^2 \sin mx\,[\alpha_n \sinh my + \beta_n(\sinh my + my\cosh my) + \gamma_n \cosh my + \delta_n(\cosh my + my\sinh my)]$$

$$F_n(y) = A_n \cosh my + B_n my \sinh my + C_n \sinh my + D_n my \cosh my, \qquad m = \frac{2n\pi}{\ell};$$

$$(S_{xx})_n = +m^2 \sin mx\,[A_n \cosh my + B_n(2\cosh my + my\sinh my) + C_n \sinh my + D_n(2\sinh my + my\cosh my)]$$

$$(S_{yy})_n = -m^2 \sin mx\,[A_n \cosh my + B_n my\sinh my + C_n \sinh my + D_n my\cosh my]$$

$$(S_{xy})_n = -m^2 \cos mx\,[A_n \sinh my + B_n(\sinh my + my\cosh my) + C_n \cosh my + D_n(\cosh my + my\sinh my)]$$

$$\sigma_{xx} = +\frac{\partial^2\phi_0}{\partial y^2} + \sum_{n=1}^{\infty}(\sigma_{xx})_n + \sum_{n=1}^{\infty}(S_{xx})_n$$

$$\sigma_{yy} = +\frac{\partial^2\phi_0}{\partial x^2} + \sum_{n=1}^{\infty}(\sigma_{yy})_n + \sum_{n=1}^{\infty}(S_{yy})_n$$

$$\sigma_{xy} = -\frac{\partial^2\phi_0}{\partial x\partial y} + \sum_{n=1}^{\infty}(\sigma_{xy})_n + \sum_{n=1}^{\infty}(S_{xy})_n$$

avec

$$\sigma_{xy}\left(x,\pm\frac{c}{2}\right)=0 \quad \text{et} \quad \sigma_{yy}\left(x,\pm\frac{c}{2}\right)=-q^{\pm}(x)\,.$$

8 Problèmes plans en coordonnées polaires

8.1 Equations générales

Déformations planes	**Contraintes planes**

8.1.1 Hypothèses

$$f_r=f_r(r,\theta)\,,\qquad f_\theta=f_\theta(r,\theta)\,,\qquad f_z=0;\qquad u_r=u_r(r,\theta)\,,\qquad u_\theta=u_\theta(r,\theta)\,,$$

Déformations planes :

$$u_z=0,\qquad \varepsilon_{zz}=\varepsilon_{rz}=\varepsilon_{\theta z}=0.$$

Contraintes planes :

$$\sigma_{zz}=\sigma_{rz}=\sigma_{\theta z}=0.$$

8.1.2 Equations d'équilibre

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+f_r=0,\qquad \frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\sigma_{r\theta}}{r}+f_\theta=0,$$

8.1.3 Equation de compatibilité

Déformations planes :

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\sigma_{rr}+\sigma_{\theta\theta}) =-\frac{1}{1-\nu}\left(\frac{\partial f_r}{\partial r}+\frac{1}{r}\left(f_r+\frac{\partial f_\theta}{\theta}\right)\right)$$

Contraintes planes :

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\sigma_{rr}+\sigma_{\theta\theta}) =-(1+\nu)\left(\frac{\partial f_r}{\partial r}+\frac{1}{r}\left(f_r+\frac{\partial f_\theta}{\theta}\right)\right)$$

8.1.4 Relations déformations–déplacements

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\qquad \varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta},\qquad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\qquad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$$

8.2 Fonction de contraintes d'Airy $\phi(r,\theta)$

8.2.1 Hypothèse supplémentaire

$$f_r=f_\theta=0$$

8.2.2 Equations d'équilibre

$$\sigma_{rr}=+\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial\phi}{\partial r}\right),\qquad \sigma_{\theta\theta}=+\frac{\partial^2\phi}{\partial r^2},$$
$$\sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=-\frac{1}{r}\left(\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).$$

8.3 Equation de compatibilité

$$\Delta\Delta\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0.$$

8.3.1 C.L. statiques

$$\begin{aligned} \sigma_{rr}n_r + \sigma_{r\theta}n_\theta &= \bar{t}_r, \\ \sigma_{r\theta}n_r + \sigma_{\theta\theta}n_\theta &= \bar{t}_\theta \end{aligned} \qquad \text{sur } \partial\Omega_F$$

8.3.2 C.L. cinématiques

$$u_r = \bar{u}_r, \qquad u_\theta = \bar{u}_\theta \qquad \text{sur } \partial\Omega_U$$

8.3.3 Problèmes plans axisymétriques avec $\phi = \phi(r)$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 \phi = 0 \quad \Longrightarrow \quad \phi(r) = A\ln r + Br^2\ln r + Cr^2 + D$$

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr} = +\frac{A}{r^2} + B(1 + 2\ln r) + 2C, \qquad \sigma_{\theta\theta} = \frac{d^2\phi}{dr^2} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C, \qquad \sigma_{r\theta} = 0$$

$$\{r = 0\} \in \Omega \quad \Longrightarrow \quad A = B = 0 \qquad \text{(compatibilité)}$$

Contraintes planes

$$\begin{aligned}
\varepsilon_{rr} &= \frac{1}{E}\left\{+\frac{(1+\nu)A}{r^2} + 2(1-\nu)B\ln r + (1-3\nu)B + 2(1-\nu)C\right\} \\
\varepsilon_{\theta\theta} &= \frac{1}{E}\left\{-\frac{(1+\nu)A}{r^2} + 2(1-\nu)B\ln r + (3-\nu)B + 2(1-\nu)C\right\} \\
\varepsilon_{zz} &= -\frac{\nu}{1-\nu}(\varepsilon_{rr} + \varepsilon_{\theta\theta}) \\
\varepsilon_{r\theta} &= \varepsilon_{rz} = \varepsilon_{\theta z} = 0
\end{aligned}$$

$$\begin{aligned}
u_r &= \frac{1}{E}\left\{-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\ln r - (1+\nu)Br + 2(1-\nu)Cr\right\} + k_1\cos\theta + k_2\sin\theta \\
u_\theta &= \frac{4}{E}Br\theta + k_2\cos\theta - k_1\sin\theta + k_3 r
\end{aligned}$$

8.3.4 Fonctions d'Airy périodiques

$$\phi(r,\theta) = f_0(r) + \sum_{m=1}^{\infty} f_m(r)\cos m\theta + \sum_{m=1}^{\infty} F_m(r)\sin m\theta$$

$$
\begin{cases}
f_0 = \alpha_0 \ln r + \beta_0 r^2 \ln r + \gamma_0 r^2 + \delta_0 \\
f_1 = \alpha_1 r + \beta_1 r \ln r + \gamma_1 \dfrac{1}{r} + \delta_1 r^3 \\
f_m = \alpha_m r^{+m} + \beta_m r^{-m} + \gamma_m r^{2+m} + \delta_m r^{2-m} & m > 1 \\
F_1 = A_1 r + B_1 r \ln r + C_1 \dfrac{1}{r} + D_1 r^3 \\
F_m = A_m r^{+m} + B_m r^{-m} + C_1 r^{2+m} + D_m r^{2-m} & m > 1
\end{cases}
$$

9 Thermoélasticité

9.1 Problème thermique

9.1.1 Equation de la chaleur et loi de Fourier

$$
-\frac{\partial q_i}{\partial x_i} + r = \rho c \dot{T}; \qquad -q_i = k \frac{\partial T}{\partial x_i}.
$$

9.1.2 Conditions aux limites et initiales

$$
T = \bar{T} \quad \text{sur } \Gamma_T, \qquad -q_i n_i = \bar{q} \quad \text{sur } \Gamma_q; \qquad T(\boldsymbol{x}, 0) = T_0(\boldsymbol{x}) \quad \text{dans } \Omega
$$

9.2 Thermoélasticité linéaire isotrope

$$
\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{mm}\delta_{ij} + \alpha\left(T - T_{\text{ref}}\right)\delta_{ij}
$$

$$
\sigma_{ij} = \frac{E\nu}{(1-2\nu)(1+\nu)}\varepsilon_{mm}\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij} - \frac{E}{1-2\nu}\alpha\left(T - T_{\text{ref}}\right)\delta_{ij}
$$

10 Stabilité des équilibres

10.1 Méthode directe

Equilibre stable $\Longleftrightarrow$ deséquilibre dans toute déviation $v(x)$ non triviale de la configuration initiale.

10.2 Méthode énergétique

10.2.1 Théorème

$$
\text{Equilibre stable} \quad \Longleftrightarrow \quad \delta E = \delta W - \delta \mathcal{T} > 0 \quad \forall\, \delta v \text{ C.A.}
$$

$$
\begin{aligned}
\delta v &= \text{déplacement virtuel cinématiquement admissible,} \\
\delta W &= \text{variation du potentiel intérieur sous } \delta v, \\
\delta \mathcal{T} &= \text{travail des forces extérieures sous } \delta v.
\end{aligned}
$$

10.2.2 Méthode approchée

$$\delta W \simeq \frac{1}{2}\int_0^\ell \frac{M^2}{EI}\,dx \simeq \frac{1}{2}\int_0^\ell EI\left(\frac{d^2v}{dx^2}\right)^2 dx; \qquad \delta\mathcal{T} \quad \simeq P\cdot\delta u_P; \qquad \delta u_P \simeq \frac{1}{2}\int_0^\ell \left(\frac{dv}{dx}\right)^2 dx$$

$$P_c = \inf_{v\text{C.A.}} \frac{\delta W\left(v''\right)}{\delta u_P\left(v'\right)} \simeq \frac{\delta W\left(\tilde{v}''\right)}{\delta u_P\left(\tilde{v}'\right)}$$

pour une forme de flambement présumée $\tilde{v}\,(x)$.

A Calculs en coordonnées cylindriques

A.1 Changements de base

$\boldsymbol{v}$ étant un vecteur, $\boldsymbol{\sigma}$ un tenseur *symétrique* d'ordre 2 *quelconque*, $\{\boldsymbol{e}_i\}$ et $\{\boldsymbol{E}_i\}$ deux bases orthonormées :

$$\begin{array}{cccc}
\boldsymbol{E}_i = Q_{ik}\boldsymbol{e}_k, & \boldsymbol{e}_i = Q_{ki}\boldsymbol{E}_k, & Q_{ij} = \boldsymbol{E}_i.\boldsymbol{e}_j, & \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I} \\
V_i = Q_{ik}v_k, & v_i = Q_{ki}V_k, & \boldsymbol{V} = \boldsymbol{Q}\boldsymbol{v}, & \boldsymbol{v} = \boldsymbol{Q}^T\boldsymbol{V} \\
\Sigma_{ij} = Q_{ik}Q_{jl}\sigma_{kl}, & \sigma_{ij} = Q_{ki}Q_{lj}\Sigma_{kl}, & \boldsymbol{\Sigma} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T, & \boldsymbol{\sigma} = \boldsymbol{Q}^T\boldsymbol{\Sigma}\boldsymbol{Q}
\end{array}$$

A.2 Coordonnées cylindriques

$$\begin{aligned}
\boldsymbol{e}_r &= \quad \cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y, \\
\boldsymbol{e}_\theta &= -\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_y
\end{aligned}
\qquad\qquad
\boldsymbol{Q} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V_r = v_x\cos\theta + v_y\sin\theta, \qquad V_\theta = -v_x\sin\theta + v_y\cos\theta, \qquad V_z = v_z$$

$$\begin{aligned}
\sigma_{rr} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \\
&= \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) + \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta + \sigma_{xy}\sin 2\theta \\
\sigma_{\theta\theta} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \\
&= \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) - \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta - \sigma_{xy}\sin 2\theta \\
\sigma_{r\theta} &= -\left(\sigma_{xx} - \sigma_{yy}\right)\sin\theta\cos\theta + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \\
&= -\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta + \sigma_{xy}\cos 2\theta \\
\sigma_{rz} &= \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta \\
\sigma_{\theta z} &= -\sigma_{xz}\sin\theta + \sigma_{yz}\cos\theta
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx} &= \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - 2\sigma_{r\theta}\sin\theta\cos\theta \\
\sigma_{yy} &= \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + 2\sigma_{r\theta}\sin\theta\cos\theta \\
\sigma_{xy} &= \left(\sigma_{rr} - \sigma_{\theta\theta}\right)\sin\theta\cos\theta + \sigma_{r\theta}\left(\cos^2\theta - \sin^2\theta\right) \\
\sigma_{xz} &= \sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta \\
\sigma_{yz} &= \sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta
\end{aligned}$$

$$\mathbf{grad}\, g = \begin{bmatrix} \dfrac{\partial g}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial g}{\partial \theta} \\ \dfrac{\partial g}{\partial z} \end{bmatrix}_{(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)}$$

$$\mathbf{grad}\,\mathbf{grad}\, g = \begin{bmatrix} \dfrac{\partial^2 g}{\partial r^2} & \dfrac{1}{r}\left(\dfrac{\partial^2 g}{\partial r \partial \theta} - \dfrac{1}{r}\dfrac{\partial g}{\partial \theta}\right) = \dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial g}{\partial \theta}\right) & \dfrac{\partial^2 g}{\partial r \partial z} \\ \text{sym} & \dfrac{1}{r}\left(\dfrac{1}{r}\dfrac{\partial^2 g}{\partial \theta^2} + \dfrac{\partial g}{\partial r}\right) & \dfrac{1}{r}\dfrac{\partial^2 g}{\partial \theta \partial z} \\ \text{sym} & \text{sym} & \dfrac{\partial^2 g}{\partial z^2} \end{bmatrix}_{(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)}$$

$$\Delta g = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2}$$

$$\Delta\Delta g = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Delta g}{\partial \theta^2} + \frac{\partial^2 \Delta g}{\partial z^2}$$

$$\mathbf{grad}\,\boldsymbol{f} = \begin{bmatrix} \dfrac{\partial f_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial f_r}{\partial \theta} - f_\theta\right) & \dfrac{\partial f_r}{\partial z} \\ \dfrac{\partial f_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial f_\theta}{\partial \theta} + f_r\right) & \dfrac{\partial f_\theta}{\partial z} \\ \dfrac{\partial f_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial f_z}{\partial \theta} & \dfrac{\partial f_z}{\partial z} \end{bmatrix}_{(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)}$$

$$\operatorname{div}\boldsymbol{f} = \frac{\partial f_r}{\partial r} + \frac{1}{r}\left(\frac{\partial f_\theta}{\partial \theta} + f_r\right) + \frac{\partial f_z}{\partial z}$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\varepsilon_{zr} = \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial r}\right), \qquad \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + f_\theta &= 0 \\ \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z &= 0 \end{aligned}$$