ON PFAFF'S EQUATIONS OF MOTION IN DYNAMICS; APPLICATIONS TO SATELLITE THEORY

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Abstract. In this article we study a form of equations of motion which is different from Lagrange's and Hamilton's equations: Pfaff's equations of motion. Pfaff's equations of motion were published in 1815 and are remarkably elegant as well as general, but still they are much less well known. Pfaff's equations can also be considered as the Euler–Lagrange equations derived from the linear Lagrangian rather than the usual Lagrangian which is quadratic in the velocity components. The article first treats the theory of changes of variables in Pfaff's equations and the connections with canonical equations as well as canonical transformations. Then the applications to the perturbed two-body problem are treated in detail. Finally, the Pfaffians are given in Hill variables and Scheifele variables. With these two sets of variables, the use of the true anomaly as independent variable is also considered.

1. Introduction

The theory of Integral Invariants of Poincaré (1892, 1893, 1899) and Cartan (1958) offers a very fundamental method in the study of dynamics. In particular, there are two important concepts: first, there is the Pfaffian or differential form (a 1-form) which is also Cartan's complete relative Integral Invariant, and then there is the 2-form, the bilinear covariant which is Cartan's absolute Integral Invariant. Both forms can be used to derive and transform the equations of motion in dynamics in an easy way. In fact, the 2-form offers the most elegant formulation of dynamics.

In this article we will present the applications of the Pfaffian or 1-form to the formulation of equations of motion in dynamics. We intentionally limit our notations to linear differential forms and we avoid the use of 2-forms as well as the notations of exterior differentiation. We agree that this results in some loss of elegance and efficiency; the reason for doing this is to cast the theory in a form which is usable by aerospace engineers and specialists in space research, who often are not trained to use exterior differentiation. The price we have to pay for this concession is that some derivations are lengthier. However, they only need the concepts of standard vector calculus and in particular the gradient and the curl, which are taught in the undergraduate programs in engineering. For those readers who want to study the more advanced aspects of differential forms and in particular the use of exterior differential calculus, we refer to the works of Flanders (1963), Choquet-Bruhat (1968), Losco (1972, 1974), Bryant (1977), and Abraham (1967), besides the fundamental work of Cartan himself (1958).

The theory of dynamics that we will develop will be entirely founded on a Pfaffian or linear differential form. This will result in equations of motion which we call Pfaff's equations. These equations are different in form but equivalent to Lagrange's

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and Hamilton's equations. Pfaff's equations (1815) were derived shortly after Lagrange's but before Hamilton's equations. They offer more flexibility than both Lagrange's and Hamilton's but they are less well known. Birkhoff (1927, p. 55) indicates the remarkable properties of Pfaff's equations and some other classical books usually spend about a page on them (Whittaker, 1959, Sections 127 and 137; Greenwood, 1977, p. 247).

A few important recent articles have developed the applications of Pfaff's equations of motion in celestial mechanics (Bilimovitch, 1942; Musen, 1964; Langlois, 1968).

We will extend here some of these works and we especially stress the possibility of making non-canonical (and canonical) transformations with the Pfaffian. For a system with n degrees of freedom, we will use a phase space with $2n + 1$ dimensions, considering the time t as one of the variables playing an identical role to the coordinates and momenta.

At the end of the paper we will show the applications to celestial mechanics and to the perturbed two-body problem (satellite theory). In the last sections we will show some sets of variables which would be particularly well-suited for the use of the true anomaly as independent variable.

2. Definitions

In what follows we will consider an extended phase space with $m = 2n + 1$ dimensions, where n is the number of degrees of freedom of the dynamical system. The additional variable is the time t which will be considered similar to the $2n$ variables. In other words, all $(2n + 1)$ variables will play the same role. Let us have $m = 2n + 1$ variables x^i and m functions $X_i(x^j)$ of these variables. The vector

$$\mathbf{X} = (X_1, X_2, \ldots, X_m) \tag{1}$$

will be called Pfaff's vector of the system. The dot product of $\mathbf{X}$ with $\mathbf{dx} = (\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^m)$ will be called a Pfaff's expression (Pfaff, 1815) or, shorter, a 'Pfaffian' Φ:

$$\Phi = \mathbf{X} \cdot \mathbf{dx} = X_i \, \mathrm{d}x^i. \tag{2}$$

We will also make frequent use of the curl of the vector $\mathbf{X}$. This is a skew-symmetric tensor (twice co-variant):

$$a_{ij} = \operatorname{curl} \mathbf{X} = \frac{\partial X_i}{\partial x^j} - \frac{\partial X_j}{\partial x^i} = A - A^T. \tag{3}$$

The matrix A is the matrix with partial derivatives of Pfaff's vector $\mathbf{X}$; in other words, the gradient of $\mathbf{X}$. Another important expression associated with the Pfaff vector $\mathbf{X}$ is the *bilinear covariant*. It is the scalar expression

$$C = \sum_{i,j}^{m} \left(\frac{\partial X_i}{\partial x^j} - \frac{\partial X_j}{\partial x^i} \right) \mathrm{d}x^i \delta x^j, \tag{4}$$

where $\mathbf{dx} = (\mathrm{d}x^i)$ and $\delta\mathbf{x} = (\delta x^i)$ indicate two different differentials. The Pfaffian and the bilinear covariant are closely related to the theory of integral invariants but we are not interested in this aspect here.

An extremely important system of $m = 2n + 1$ first-order differential equations associated with the Pfaffian Φ and Pfaff's vector $\mathbf{X}$ is "*Pfaff's first system of equations*" or the "*associated Pfaff system*":

$$\sum_{j=1}^{m} a_{ij}\,\mathrm{d}x^j = \sum_{j=1}^{m} \left(\frac{\partial X_i}{\partial x^j} - \frac{\partial X_j}{\partial x^i}\right) \mathrm{d}x^j = 0. \tag{5}$$

We will see that the associated Pfaff system is the system of equations of motion of a dynamical system. The associated Pfaff system (5) may also be written in several other forms. For instance, in matrix form we may write it as:

$$(A^T - A)\,\mathrm{d}\mathbf{x} = (\mathrm{curl}\,\mathbf{X})\,\mathrm{d}\mathbf{x} = 0, \tag{6}$$

where $A^T - A = \mathrm{curl}\,\mathbf{X}$ is the $(2n+1) \times (2n+1)$-matrix defined before, and $\mathrm{d}\mathbf{x}$, a $(2n+1)$-vector. Another short symbolic form of the associated Pfaff system is:

$$\frac{\partial X_i}{\partial \mathbf{x}} \cdot \mathrm{d}\mathbf{x} - \frac{\partial}{\partial x^i}(\mathbf{X} \cdot \mathrm{d}\mathbf{x}) = 0 \tag{7}$$

or

$$(\mathrm{grad}\,\mathbf{X}) \cdot \mathrm{d}\mathbf{x} - \mathrm{grad}\,\Phi = 0, \tag{8}$$

or even

$$\mathrm{d}\mathbf{X} - \mathrm{grad}\,\Phi = 0. \tag{9}$$

It is known that the condition under which the Pfaffian (2) is an exact differential is that the curl of $\mathbf{X}$ be zero; in other words, that $A - A^T$ be zero (or that A be symmetric). As a consequence, if an exact differential is added to a Pfaffian, the associated Pfaff system will be unchanged. It is also noted that the Pfaff system (5) is the set of Euler–Lagrange equations obtained by extremalizing the integral of the linear Lagrangian $\Phi/\mathrm{d}t$ in $2n$ coordinates.

3. Changes of Variables in a Pfaffian

We will now prove the property which makes the Pfaffian theory so useful: the Pfaffian and the associated Pfaff system do not change in form when a change of coordinates is made from $\mathbf{x}$ to $\mathbf{y}$ in Pfaff's phase space (with dimension $2n + 1$). To show this important fact, let us define a change of variables from $\mathbf{x}$ $(= x^i)$ to $\mathbf{y}$ $(= y^j)$. In this section we make implicit usage of Einstein's summation convention; we have then $\mathbf{x}(\mathbf{y})$ and

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial x^j}\,\mathrm{d}y^j. \tag{10}$$

The transformation of Pfaff's vector $\mathbf{X}$ is obtained by considering that the Pfaffian itself (Φ) is a scalar and thus invariant:

$$\Phi = X_i \, \mathrm{d}x^i = Y_i \, \mathrm{d}y^i = X_i \frac{\partial x^i}{\partial y^j} \mathrm{d}y^j. \tag{11}$$

Consequently,

$$Y_i = X_j \frac{\partial x^j}{\partial y^i} \, . \tag{12}$$

Let us now also see how the associated Pfaff system transforms under the same change of variables. Starting by taking derivatives of (12) and multiplying by $\mathrm{d}y^k$, we find:

$$\left(\frac{\partial Y_i}{\partial y^k} - \frac{\partial Y_k}{\partial y^i}\right) \mathrm{d}y^k = \frac{\partial x^j}{\partial y^i}\left[\left(\frac{\partial X_j}{\partial x^l} - \frac{\partial X_l}{\partial x^j}\right)\mathrm{d}x^l\right] \cdot \tag{13}$$

This last formula thus shows that if we have the associated Pfaff system in the **x**-coordinates, the right-hand side of (13) will be zero, and we will also have the associated Pfaff system in the **y**-coordinates:

$$\left(\frac{\partial Y_i}{\partial y^k} - \frac{\partial Y_k}{\partial y^i}\right) \mathrm{d}y^k = 0. \tag{14}$$

Several remarks may now be made about the demonstration that has just been given. First of all, the result (14) about the invariance of the form of the associated Pfaff system was to be expected. This is because all the ingredients of the associated Pfaff system are *tensors*. It is well known that if the equations are of a tensorial character, then their form is independent of the coordinate system that is used.

The second remark will be of equal importance. In the previous manipulations, we have made no usage at all of the number of components of the **y**-vector (the dimension of the new Pfaff's phase space!). It is seen that it thus is possible to make a change of variables from **x** to **y**, which introduces a *larger number of variables*. If this is the case, the new variables will be related by some constraints, but these constraints are not used in the derivation of the new Pfaffian and the associated system. They may be used, however, to simplify the associated Pfaff system. On the other hand it will not be permissible, in general, to make a change of variables in the Pfaffian which *decreases* the number of variables or components. In most cases, however, the number of variables will be the same for **x** and **y**, and if the matrix $\partial\mathbf{x}/\partial\mathbf{y}$ is not singular, the transformation can be made in both directions.

4. The Pfaffian of a Canonical System

Let us now take a dynamical system represented by canonical equations and a Hamiltonian $H(q, p, t)$. To simplify the writings we will assume that the number of degrees of freedom is one.

$$\frac{\mathrm{d}q}{\mathrm{d}t} = +\frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} \, . \tag{15}$$

Pfaff's phase space will thus be of dimension 3 and we will take the three coordinates to be $\mathbf{x} = (q, p, t)$. We will also take the Pfaff vector defined by $\mathbf{X} = (p, 0, -E)$ where $E = H$ is the energy of the system. In the present case, we may also write

$$\mathbf{X} = (p, 0, -H). \tag{16}$$

The Pfaffian Φ will thus be in this case

$$\Phi = p\,\mathrm{d}q - H\,\mathrm{d}t. \tag{17}$$

Let us now show that the associated Pfaff system is equivalent to the equations of motion (15). In other words, the associated Pfaff system is the system of equations of motion of the problem.

First of all, we have the partial derivatives

$$A = \left(\frac{\partial X_i}{\partial x^j}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\dfrac{\partial H}{\partial q} & -\dfrac{\partial H}{\partial j} & -\dfrac{\partial H}{\partial t} \end{bmatrix}. \tag{18}$$

The skew-symmetric matrix $A - A^T$ is thus

$$\operatorname{curl} \mathbf{X} = A - A^T = \begin{bmatrix} 0 & 1 & \dfrac{\partial H}{\partial q} \\ -1 & 0 & \dfrac{\partial H}{\partial p} \\ -\dfrac{\partial H}{\partial q} & -\dfrac{\partial H}{\partial p} & 0 \end{bmatrix}. \tag{19}$$

The associated first Pfaff system is now obtained by multiplying the previous matrix at the right-side by a column-vector with components ($\mathrm{d}q$, $\mathrm{d}p$, $\mathrm{d}t$):

$$\begin{cases} \mathrm{d}p + \dfrac{\partial H}{\partial q}\,\mathrm{d}t = 0, \\ -\mathrm{d}q + \dfrac{\partial H}{\partial p}\,\mathrm{d}t = 0, \\ -\dfrac{\partial H}{\partial q}\,\mathrm{d}q - \dfrac{\partial H}{\partial p}\,\mathrm{d}p = 0. \end{cases} \tag{20}$$

These equations are readily seen to be the same as the canonical equations (15).

The generalization to more than one degree of freedom is obvious. If we assume n degrees of freedom, we have the Pfaff's vector

$$\mathbf{X} = (p_1, p_2, \ldots, p_n, 0, 0, \ldots, 0, -H) \tag{21}$$

and the Pfaffian

$$\Phi = p_i\,\mathrm{d}q^i - H\,\mathrm{d}t. \tag{22}$$

The $2n + 1$ variables $\mathbf{x}$ are in the following order:

$$\mathbf{x} = (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n, t). \tag{23}$$

Now that a Pfaffian is available in canonical variables, we can make a change of variables to some new ones which may be canonical or not. We will be assured that in any new system of variables the equations of motion will be the first associated Pfaff system.

5. Canonical Transformations Based on the Pfaffian

We will take the Pfaffian (22) as a starting point for the introduction of a theory of canonical transformations. A transformation from the variables (q, p) to the new variables (Q, P) is said to be canonical if the general form of the Pfaffian Φ is unchanged. Thus, the new Pfaffian is

$$\Phi = P_i \, \mathrm{d}Q^i - K \, \mathrm{d}t \tag{24}$$

where K is the new Hamiltonian.

The transformation from the old to the new variables is made by adding an exact differential to the Pfaffian and by appropriately grouping the different terms. We will show here a small example related to the Poincaré transformation, but we will use the technique extensively in the following sections to manipulate the Pfaffian of the two-body problem.

To illustrate the method, let us start from the expression $P \, \mathrm{d}Q$ and subtract half the differential of PQ:

$$P \, \mathrm{d}Q - \tfrac{1}{2} \mathrm{d}(PQ) = \tfrac{1}{2}(P \, \mathrm{d}Q - Q \, \mathrm{d}P). \tag{25}$$

By manipulating this result into

$$\frac{P^2 + Q^2}{2} \cdot \frac{P \, \mathrm{d}Q - Q \, \mathrm{d}P}{P^2 + Q^2} = \frac{P^2 + Q^2}{2} \, \mathrm{d}\left[\arctan \frac{Q}{P}\right], \tag{26}$$

we see that it can be written in the form $p \, \mathrm{d}q$ and that the following transformation is canonical:

$$p = (P^2 + Q^2)/2$$
$$q = \arctan (Q/P). \tag{27}$$

This transformation is equivalent to

$$P = \sqrt{2p} \, \cos q$$
$$Q = \sqrt{2p} \sin q. \tag{28}$$

It is the well known Poincaré transformation.

Let us note that the resulting transformation is a function not only of the exact differential but of the grouping of the terms as well. For instance, the above expression (26) can also be written as

$$\frac{P^2}{2}\frac{P\,\mathrm{d}Q - Q\,\mathrm{d}P}{P^2} = \frac{P^2}{2}\,\mathrm{d}\left(\frac{Q}{p}\right). \tag{29}$$

This shows us the existence of a new canonical transformation

$$p = P^2/2; \quad q = Q/P. \tag{30}$$

It can be shown that the exact differential which has to be added to the Pfaffian is equal to the standard generating function of the first type, expressed exclusively in terms of old variables.

6. Different Forms of the Pfaffian in Dynamics

If the variables which are used are not canonical, the Pfaffians of dynamics can be written in a large number of different forms. For instance, if rectangular coordinates are used to represent the state of a system of N particles m_i, the Pfaffian may be written as

$$\Phi = \sum_{i=1}^{m} m_i \mathbf{v}_i \cdot \mathrm{d}\mathbf{x}_i - E\,\mathrm{d}t. \tag{31}$$

There are six coordinates defining the position and velocity of each particle and we have $6N + 1$ Pfaff equations. The symbol E represents the total energy of the system:

$$E = \sum_{i=1}^{N} \tfrac{1}{2} m_i \mathbf{v}_i^2 - U = T - U. \tag{32}$$

If the system is described by generalized coordinates q^i and a given Lagrangian

$$\mathscr{L} = \tfrac{1}{2} g_{ij}\,\dot{q}^i \dot{q}^j + A_i \dot{q}^i + U. \tag{33}$$

it is then also very easy to construct the Pfaffian Φ of the system. The momenta p_i and the energy E are given by

$$p_i = g_{ij}\dot{q}^j + A_i \tag{34a}$$

$$E = \tfrac{1}{2} g_{ij}\dot{q}^i \dot{q}^j - U. \tag{34b}$$

The Pfaffian may be considered as

$$\Phi = (g_{ij}\dot{q}^i + A_j)\,\mathrm{d}q^j - (\tfrac{1}{2} g_{ij}\dot{q}^i\dot{q}^j - U)\,\mathrm{d}t. \tag{35}$$

The Pfaff's equations derived from it are equivalent to the energy equation and the classical Euler–Lagrange equation of motion:

$$g_{ik}\ddot{q}^k + \Gamma_{jki}\dot{q}^j\dot{q}^k + A_{ik}\dot{q}^k - U_i = 0 \tag{36}$$

where the standard definitions have to be used for the coefficients:

$$A_{ik} = \frac{\partial A_i}{\partial q^k} - \frac{\partial A_k}{\partial q^i},$$

$$\Gamma_{jki} = \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right]. \tag{37}$$

The important advantage of the Pfaffian formulation over the Lagrangian formulation resides in the following facts. In the Lagrangian equations we can only make changes of coordinates in the n-dimensional configuration space, of the general form

$$q^i = q^i(u^j, t), \qquad \dot{q}^i = \frac{\partial q^i}{\partial u^j}\dot{u}^j + \frac{\partial q^i}{\partial t}. \tag{38}$$

In other words, we are restricted by the constraint that n variables must be the time-derivatives of n of the other variables. In the Hamiltonian formulation, we do not have this restriction, and we work in a $2n$-diemensional phase space. However, we have the restriction that all changes of variables must be canonical. In the Pfaffian formulation, both restrictions will disappear. We can here make changes of coordinates of the form

$$q^i = f^i(u^j, v^j, t), \qquad \dot{q}^i = g^i(u^j, v^j, t), \tag{39}$$

where f^i and g^i are $2n$ given arbitrary functions of the new coordinates and time. It is clear that the $2n$ new variables u and v introduced in (39) are related by n constraint equations, but these constraints will automatically be satisfied by Pfaff's $(2n + 1)$ equations of motion. Another advantage can be added to Pfaff's formulation: changes of independent variable t can be made simultaneously with the changes of coordinates.

7. The Pfaffian of Perturbed Two-Body Motion

The present and the following sections of this chapter will give some examples to celestial mechanics; in particular, the perturbed two-body problem. The Pfaffian of this problem is

$$\begin{aligned}\Phi &= \mathbf{v}\cdot \mathrm{d}\mathbf{x} - E_0\,\mathrm{d}t = \mathbf{v}\cdot\mathrm{d}\mathbf{x} - \left(\frac{\mathbf{v}^2}{2} - \frac{\mu}{r} - R\right)\mathrm{d}t \\ &= \mathbf{v}\cdot\mathrm{d}\mathbf{x} + \left(\frac{\mu}{2a} + R\right)\mathrm{d}t.\end{aligned} \tag{40}$$

The constant μ is the constant of gravitation and r is the length of the radius-vector. The disturbing function is represented by R. The position-vector $\mathbf{x}$ in the perturbed two-body problem is given by the well-known formulas

$$\begin{aligned}\mathbf{x} &= a\mathbf{P}(\cos E - e) + a\sqrt{1 - e^2}\,\mathbf{Q}\sin E. \\ M &= E - e\sin E.\end{aligned} \tag{41}$$

The symbols $\mathbf{P}$ and $\mathbf{Q}$ represent unit-vectors in the orbit plane; M represents the mean anomaly, and E the eccentric anomaly.

The position vector $\mathbf{x}$ in the two-body problem will be assumed to be a function of six new variables: $a_\alpha = (a, e, i, M, \omega, \Omega)$. We will then have for $\mathrm{d}\mathbf{x}$ the expression

$$\mathrm{d}\mathbf{x} = \sum_{\alpha=1}^{6} \frac{\partial x}{\partial a_\alpha}\,\mathrm{d}a_\alpha. \tag{42}$$

The partial derivatives of the two-body problem have been published elsewhere (Broucke, 1970) and are not reproduced here. Using the expression (42), we find for the Pfaffian Φ the form:

$$\Phi = \sum_{\alpha=1}^{6} \left(\mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial a_\alpha}\right) \mathrm{d}a_\alpha - E_0\,\mathrm{d}t. \tag{43}$$

Thus, the next step is to evaluate the six scalar products present in the previous formula. After some simplifications, we find the following results:

$$\begin{aligned}
&\left(\mathbf{v}\cdot\frac{\partial \mathbf{y}}{\partial a}\right) = na\sin E. && \left(\mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial M}\right) = \sqrt{\mu a^3}\left(\frac{2}{r}-\frac{1}{a}\right),\\
&\left(\mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial e}\right) = \frac{2na^3\sin E}{r}, && \left(\mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial \omega}\right) = \sqrt{\mu a(1-e^2)},\\
&\left(\mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial i}\right) = 0, && \left(\mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial \Omega}\right) = \sqrt{\mu a(1-e^2)}\cos i.
\end{aligned} \tag{44}$$

Thus Pfaffian (40) becomes thus:

$$\begin{aligned}
\Phi = {} & nae\sin E\,\mathrm{d}a + \frac{2na^3}{r}\sin E\,\mathrm{d}e + \sqrt{\mu a^3}\left(\frac{2}{r}-\frac{1}{a}\right)\mathrm{d}M + \\
& + \sqrt{\mu a(1-e^2)}(\mathrm{d}\omega + \cos i\,\mathrm{d}\Omega) - E_0\,\mathrm{d}t.
\end{aligned} \tag{45}$$

This Pfaffian can be simplified, however, by adding exact differentials to it. It turns out that the most interesting quantity to add to (45) is $-2\mathrm{d}(\mathbf{v}\cdot\mathbf{x})$, where

$$\mathbf{v}\cdot\mathbf{x} = \sqrt{\mu a}\,e\sin E = r\dot{r} = \sqrt{\mu a}\,(E - M). \tag{46}$$

The differential of this quantity is:

$$2\mathrm{d}(\mathbf{v}\cdot\mathbf{x}) = \left(e\sqrt{\frac{\mu}{a}}\,\mathrm{d}a + 2\sqrt{\mu a}\,\mathrm{d}e\right)\sin E + 2\sqrt{\mu a}\left(1-\frac{r}{a}\right)\mathrm{d}E. \tag{47}$$

The quantity $\mathrm{d}E$ can be removed from it by differentiating Kepler's equation:

$$\mathrm{d}E = \frac{a}{r}\,\mathrm{d}M + \frac{a}{r}\sin E\,\mathrm{d}e. \tag{48}$$

The previous exact differential becomes thus:

$$2\mathrm{d}(\mathbf{v}\cdot\mathbf{x}) = \left(nae\,\mathrm{d}a + \frac{2na^3}{r}\,\mathrm{d}e\right)\sin E + \left[\sqrt{\mu a^3}\left(\frac{2}{r}-\frac{1}{a}\right) - \sqrt{\mu a}\right]\mathrm{d}M. \tag{49}$$

Substracting this quantity from the Pfaffian (45) now gives the elegant result

$$\Phi = \sqrt{\mu a}\,\mathrm{d}M + \sqrt{\mu a(1-e^2)}\,\mathrm{d}\omega + \sqrt{\mu a(1-e^2)}\cos i\,\mathrm{d}\Omega - E_0\,\mathrm{d}t. \tag{50}$$

A closer inspection of this Pfaffian suggests the introduction of six new variables which turn out to be canonical variables: the well-known Delaunay elements:

$$\begin{cases} L = \sqrt{\mu a}, \\ l = M, \end{cases} \quad \begin{cases} G = \sqrt{\mu a(1 - e^2)}, \\ g = \omega, \end{cases} \quad \begin{cases} H = \sqrt{\mu a(1 - e^2)} \cos i, \\ h = \Omega. \end{cases} \tag{51}$$

The Pfaffian may be written in a shorter form:

$$\Phi = L\,\mathrm{d}l + G\,\mathrm{d}g + H\,\mathrm{d}h - E_0\,\mathrm{d}t. \tag{52}$$

8. Other Pfaffians for the Perturbed Two-body Problem

A large number of new Pfaffians can be obtained by adding different exact differentials to the Pfaffian (50), (Musen, 1969). We list here a few of these results:

$$\Phi = \sqrt{\mu a}\,\mathrm{d}M + \sqrt{\mu a(1 - e^2)}\,(\mathbf{Q} \cdot \mathrm{d}\mathbf{P}) - E_0\,\mathrm{d}t, \tag{53a}$$

$$\Phi = \sqrt{\mu a}\,\mathrm{d}M - \sqrt{\mu a(1 - e^2)}\,(\mathbf{P} \cdot \mathrm{d}\mathbf{Q}) - E_0\,\mathrm{d}t, \tag{54b}$$

$$\Phi = \sqrt{\mu a}\,\mathrm{d}M + \tfrac{1}{2}\sqrt{\mu a(1 - e^2)}\,(\mathbf{Q} \cdot \mathrm{d}\mathbf{P} - \mathbf{P} \cdot \mathrm{d}\mathbf{Q}) - E_0\,\mathrm{d}t, \tag{55c}$$

$$\Phi = \sqrt{\mu a}\,\mathrm{d}M + \sqrt{\mu a(1 - e^2)}\left[\cos^2\frac{i}{2}\,\mathrm{d}(\omega + \Omega) + \sin^2\frac{i}{2}\,\mathrm{d}(\omega - \Omega\right] - - E_0\,\mathrm{d}t. \tag{56d}$$

The last of these four Pfaffians shows the existence of another canonical set (closely related to the Delaunay elements):

$$\begin{cases} P_1 = \sqrt{\mu a}, \\ Q_1 = M, \end{cases} \quad \begin{cases} P_2 = \sqrt{\mu a(1 - e^2)} \cos^2\dfrac{i}{2}, \\ Q_2 = \omega + \Omega, \end{cases} \quad \begin{cases} P_3 = \sqrt{\mu a(1 - e^2)} \sin^2\dfrac{i}{2} \\ Q_3 = \omega - \Omega. \end{cases} \tag{57}$$

Another canonical set of elements, of pure vectorial form, has been derived from the second of the four Pfaffians above, by Musen. The six new elements are defined by

$$\begin{aligned} \mathbf{p} &= \sqrt{\mu a}\,\mathbf{P}, \\ \mathbf{q} &= \sqrt{1 - e^2}\,\mathbf{Q} - M\mathbf{P}. \end{aligned} \tag{58}$$

Let us show that this is a canonical set. We have

$$\begin{aligned} \mathrm{d}\mathbf{q} &= \mathrm{d}\sqrt{1 - e^2} \cdot \mathbf{Q} + \sqrt{1 - e^2}\,\mathrm{d}\mathbf{Q} - \mathrm{d}M \cdot \mathbf{P} - M\,\mathrm{d}\mathbf{P}, \\ \mathbf{p} \cdot \mathrm{d}\mathbf{q} &= \sqrt{\mu a}\,[\mathrm{d}\sqrt{1 - e^2}\,(\mathbf{P} \cdot \mathbf{Q}) + \sqrt{1 - e^2}\,(\mathbf{P} \cdot \mathrm{d}\mathbf{Q}) - \mathbf{P}^2\,\mathrm{d}M - \\ &\quad - (\mathbf{P} \cdot \mathrm{d}\mathbf{P})\,\mathrm{d}M]. \end{aligned} \tag{59}$$

However, we know that **P** and **Q** are orthonormal unit-vectors:

$$\mathbf{P} \cdot \mathbf{Q} = 0, \quad \mathbf{P}^2 = 1, \quad \mathbf{P} \cdot \mathrm{d}\mathbf{P} = 0. \tag{60}$$

Finally, we have

$$\mathbf{p} \cdot \mathrm{d}\mathbf{q} = \sqrt{\mu a(1 - e^2)}(\mathbf{P} \cdot \mathrm{d}\mathbf{Q}) - \sqrt{\mu a}\,\mathrm{d}M. \tag{61}$$

Comparing this with (54b) shows that a new valid Pfaffian is

$$\Phi = \mathbf{p}\cdot \mathrm{d}\mathbf{q} - K\,\mathrm{d}t. \tag{62}$$

The new Hamiltonian is $K = -E_0$ and the new equations of motion have the standard canonical form:

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = +\frac{\partial K}{\partial \mathbf{p}}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial K}{\partial \mathbf{q}}. \tag{63}$$

A large number of canonical sets of elements can still be derived by manipulating the above Pfaffians. For instance, the Poincaré elements and the Lyddane elements (1963) are easily obtained.

9. The Variation-of-parameters Method with Classical Elements

It can be shown that the associated first Pfaff's system corresponding to the Pfaffian (46) gives the well-known variation-of-parameters equations. Of course, with the Delaunay elements we know what the equations of motion are because these elements are canonical. However, if we consider that the variables in (45) are the classical variables (a, e, i, M, ω, Ω, t), then the associated first Pfaff's system will give the equations of motion with these variables, which are Lagrange's planetary equations.

Let us assume that the seven variables are in the order which has just been indicated. The Pfaff's vector is then

$$X_i = (0, 0, 0, \sqrt{\mu a}, \sqrt{\mu a(1-e^2)}, \sqrt{\mu a(1-e^2)}\cos i, -E_0). \tag{64}$$

The matrix of partial derivatives of the **X**-vector is:

$$\frac{\partial X_i}{\partial x^j} = \left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{1}{2}\sqrt{\frac{\mu}{a}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\sqrt{\frac{\mu}{a}(1-e^2)} & -l\sqrt{\frac{\mu a}{1-e^2}} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\sqrt{\frac{\mu}{a}(1-e^2)}\cos i & -l\sqrt{\frac{\mu a}{1-e^2}}\cos i & -\sqrt{\mu a(1-e^2)}\sin i & 0 & 0 & 0 & 0 \\
\hline
-\frac{\mu}{2a^2}+\frac{\partial R}{\partial a} & \frac{\partial R}{\partial e} & \frac{\partial R}{\partial i} & \frac{\partial R}{\partial M} & \frac{\partial R}{\partial \omega} & \frac{\partial R}{\partial \Omega} & \frac{\partial R}{\partial t}
\end{array}\right] \tag{65}$$

Taking the transpose of the above matrix gives the equations of motion $(A - A^T)\,\mathrm{d}\mathbf{x} = 0$.

10. The Pfaffian with Hill Variables

We return now to the fundamental Pfaffian (45) of the two-body problem which, in account of the identity (49), can also be written as:

$$\Phi = 2\,\mathrm{d}(r\dot{r}) + na^2\,\mathrm{d}M + na^2\sqrt{1 - e^2}\,(\mathrm{d}\omega + \cos i\,\mathrm{d}\Omega) - E_0\,\mathrm{d}t. \tag{66}$$

However, if we use the true anomaly v as one of the variables, the identity (49) may be written as

$$2\,\mathrm{d}(r\dot{r}) + na^2\,\mathrm{d}M = \dot{r}\,\mathrm{d}r + na\sqrt{1 - e^2}\,\mathrm{d}v, \tag{67}$$

and the fundamental Pfaffian is then

$$\Phi = \dot{r}\,\mathrm{d}r + na^2\sqrt{1 - e^2}\,(\mathrm{d}u + \cos i\,\mathrm{d}\Omega) - E_0\,\mathrm{d}t. \tag{68}$$

Here we have introduced the symbol u for the argument of the latitude $\omega + v$. The last Pfaffian shows the existence of a set of canonical variables, three of which are standard Delaunay variables and the two others being r and $\dot{r} = pr$. More precisely, the coordinates are r, u, Ω, and their conjugate momenta are $\dot{r}$, G, H. These are well known as Hill's variables (1913), which have recently been used in satellite theory (Aksnes, 1972).

With this system of variables, the energy E_0 is expressed in the form:

$$E_0 = \frac{1}{2}\left(\dot{r}^2 + \frac{G^2}{r^2}\right) - U. \tag{69}$$

The equations of motion for the six Hill variables can be derived from the Pfaffian (68) as well as the Hamiltonian (69), by the usual rules. In (68), the inclination should be expressed in terms of G and $H = G\cos i$. The equations of motion are:

$$\ddot{r} = \frac{G^2}{r^3} - \frac{\mu}{r^2} + \frac{\partial R}{\partial r}, \tag{70a}$$

$$\dot{G} = \frac{\partial R}{\partial u} = rT, \tag{70b}$$

$$\dot{H} = \frac{\partial R}{\partial \Omega}, \tag{70c}$$

$$\dot{\Omega} = \frac{r\sin u}{G\sin i}\,W, \tag{70d}$$

$$\dot{u} = \frac{G}{r^2} - \frac{r\sin u\cos i}{G\sin i}\,W. \tag{70c}$$

In these equations, R represents the disturbing function ($= U - \mu/r$), while T is the transverse component of the perturbation and W the component perpendicular to the plane of the motion.

The previous equations of motion (70) would be especially easy to use with true anomaly v as an independent variable. The change of variable is made according to the differential relation

$$dt = \frac{r^2}{G}\,dv. \tag{71}$$

This equation uses the transformation factor r^2/G which is a function only of two of the Hill variables. This is the fact that makes the Hill variables so convenient in terms of the true anomaly. The transformation can be performed directly on the equations of motion (70) or on the Pfaffian (68). The new Pfaffian will be

$$\Phi = \dot{r}\,dr + G\,du + H\,d\Omega - \frac{r^2}{G}(E_0 - E_n)\,dv. \tag{72}$$

This shows that, even with the true anomaly as independent variable, the system remains canonical and the standard perturbation methods with Lie series (Hori, 1966; Deprit, 1969) can be used. However, the new Hamiltonian is

$$\frac{r^2}{G}(E_0 - E_n) = \frac{1}{2}\left(\frac{r^2\dot{r}^2}{G} + G\right) - \frac{\mu r}{G} - \frac{r^2}{G}(R + E_n), \tag{73}$$

where E_n is a numerical constant representing the total value of the energy of the system. In the next section, we will explain another approach to the use of the true anomaly.

In a final remark on the Hill variables, we would like to mention a possible extension to a larger number of dimensions with the use of redundant variables, as was mentioned at the end of Section 3 of this article. The details of the redundant variable theory were described previously by Broucke (1971, 1975). According to the previous equations (70d) and (70e), we have

$$r^2(\dot{u} + \dot{\Omega}\cos i) = G = na^2\sqrt{1 - e^2} \tag{74}$$

and it can be shown that this relation may be used to transform (68) into a new Pfaffian with nine, rather than 7 variables:

$$\Phi = \dot{r}\,dr + r^2(\dot{u} + \dot{\Omega}\cos i)(du + \cos i\,d\Omega) - E_0\,dt. \tag{75}$$

The coordinate-vector has the following nine components:

$$\mathbf{x} = (r, u, \Omega, i, \dot{r}, \dot{u}, \dot{\Omega}, \dot{i}, t). \tag{76}$$

The reader will have no difficulty constructing the Pfaff vector **X**, its curl, and finally the eight corresponding equations of motion. In the present system of variables, the position is described by the radius vector r and three Euler angles (u, Ω, i).

11. The Pfaffian with Scheifele Variables

In the previous section we have shown several interesting properties of Hill's variables because of their possible application in satellite theories based on the true anomaly as independent variable. The change of independent variable can easily be done without introducing the concept of phase space. The perturbation methods of Hori based on Lie series, can be used with the Hill variables, although the Von Zeipel method cannot be used as well. This is because the unperturbed Hamiltonian contains a coordinate r, besides the two momenta $\dot{r}$ and G.

At this moment we want to contrast the Hill variables with the Scheifele variables which also use the true anomaly (or the eccentric anomaly) as independent variable (Scheifele, 1970; Bond and Broucke, 1977). In the Scheifele theory the extended phase space plays a basic role. The resulting unperturbed Hamiltonian contains two moments only, and no coordinates, so that the standard Von Zeipel techniques are directly applicable.

Before we present the Pfaffian in Scheifele variables, we want to mention another factor which plays a role here, as well as in many other satellite theories: The Scheifele variables turn out to be very close to the Delaunay elements. A remarkable result in relation to Delaunay elements (Section 7) was that we had to add the exact differential of $na^2(E - M)$, the difference of two anomalies (Equation 46), to the Pfaffian. It turns out that in order to derive the Scheifele variables from the Delaunay variables, we will add a similar exact differential to the Pfaffian: the differential of the equation of the center, or true minus mean anomaly.

The Scheifele variables form a system of eight canonical elements, four of which are standard Delaunay elements g, h, G, H. In order to simplify the writings, we do not include these elements in the following Pfaffians. But we have first to introduce two new elements and the extended phase space concept. The two new elements are λ, equal to the time t, and its conjugate momentum Λ, equal to minus the total energy of the system. Neglecting the four above mentioned Delaunay elements, the Pfaffian can be written as

$$\Phi = \Lambda\,\mathrm{d}\lambda + L\,\mathrm{d}l - \left(\Lambda - \frac{\mu^2}{2L^2} - R\right)\mathrm{d}t. \tag{77}$$

It is well known that the numerical value of the new Hamiltonian is zero (Zare and Szebehely, 1975). In order to change to another independent variable τ, defined by

$$\mathrm{d}t = f\ \mathrm{d}\tau, \tag{78}$$

we have to replace the above Pfaffian (77) with the new one

$$\Phi = \Lambda\,\mathrm{d}\lambda + L\,\mathrm{d}l - \left(\Lambda - \frac{\mu^2}{2L^2} - R\right)f\,\mathrm{d}\tau. \tag{79}$$

Now we add an appropriate exact differential to (79), replacing $\Lambda\,\mathrm{d}\lambda + L\,\mathrm{d}l$ by a new expression:

$$\bar{\omega} = \Lambda\,\mathrm{d}\lambda + L\,\mathrm{d}l - \tfrac{3}{2}\mathrm{d}[L(l - \psi)], \tag{80}$$

where ψ is an undetermined function of λ only. The expression $\bar{\omega}$ can be put in the form

$$\bar{\omega} = \frac{\mu^2}{2L^2}\,\mathrm{d}\left[\lambda - \frac{L^3}{\mu^2}(l - \psi)\right] + \left[L + \left(\Lambda - \frac{\mu^2}{2L^2}\right)\bigg/\frac{\partial\psi}{\partial\lambda}\right]\frac{\partial\psi}{\partial\lambda}\,\mathrm{d}\lambda, \tag{81}$$

which suggests the introduction of the Scheifele variables:

$$L_s = \frac{\mu^2}{2L^2}, \tag{82a}$$

$$\Phi_s = L + \left(\Lambda - \frac{\mu^2}{2L^2}\right)\bigg/\frac{\partial\psi}{\partial\lambda}, \tag{82b}$$

$$l_s = \lambda - \frac{L^3}{\mu^2}(l - \psi), \tag{82c}$$

$$\phi_s = \psi(\lambda). \tag{82d}$$

In the present system of notations, l_s and ϕ_s are the new coordinates, while L_s and Φ_s are the corresponding canonical momenta. It is now easy to see that under the effect of the above canonical transformation, the Hamiltonian of the problem transforms into

$$E_0 = \left(\Phi_s - \frac{\mu}{(2L_s)^{1/2}}\right)\cdot f\cdot\frac{\partial\psi}{\partial\lambda} - f\cdot R. \tag{83}$$

Keeping in mind that the two functions f and ψ are still undefined, we clearly see that this Hamiltonian can be simplified drastically if we choose these functions in such a way that

$$f\cdot\frac{\partial\psi}{\partial\lambda} = +1, \tag{84a}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\mathrm{d}\psi}{\mathrm{d}\lambda} = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{f}. \tag{84b}$$

In other words, ψ should be identical to τ. On the other hand, f should be taken equal to r^2/G in order for τ to be identical with the true anomaly. In this case, the expression $l - \psi$ in (80) is the equation of the center. Some additional details on the Scheifele variables are found in Bond and Broucke (1977).

In accordance with the remark made at the end of Section 5, we can easily construct a generating function of the above canonical transformation (82). This generating function is intimately related to the exact differential introduced in (80). It is the expression

$$S_2 = (-3/2)L(l - \psi) + l_s L_s + \phi_s\Phi_s \tag{85}$$

which should be represented as a function of the old coordinates (l, λ) and new momenta (L_s, Φ_s):

$$S_2 = \frac{\mu}{(2L_s)^{1/2}}(l - \psi) + \lambda L_s + \psi\Phi_s. \tag{86}$$

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