It is therefore possible to write a Lorentz-invariant kinetic term which is first-order in the derivative,

$$\mathcal{L}_L = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \,. \tag{3.66}$$

$\psi_L^\dagger = (\psi_L^*)^T$

$\frac{\partial \psi_L^\dagger}{\partial \psi_L^*} = \frac{\partial (\psi_L^*)^T}{\partial \psi_L^*} = (1)^T$?

$1 \overset{?}{=} 1^{(T)}$?

The factor i in front is fixed by the condition that the action $\int d^4x\, \mathcal{L}_L$ is real, as we verify immediately using the fact that the matrices $\bar{\sigma}^\mu$ are hermitian. The equation of motion is obtained varying with respect to ψ_L^*, considering ψ_L^* and ψ_L as two independent fields. Since $\partial_\mu \psi_L^\dagger$ does not appear in $\mathcal{L}_L$, the Euler–Lagrange equation is simply $\partial\mathcal{L}/\partial\psi_L^* = 0$, which gives $\bar{\sigma}^\mu \partial_\mu \psi_L = 0$, or, more explicitly

$$\boxed{(\partial_0 - \sigma^i \partial_i)\psi_L = 0 \,.} \tag{3.67}$$

As a consequence (using $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ and the fact that, on a regular function, $\partial_i \partial_j$ is symmetric in i, j), $\partial_0^2 \psi_L = (\sigma^i \partial_i)(\sigma^j \partial_j)\psi_L = \partial_i^2 \psi_L$, or

$$\Box \psi_L = 0 \,. \tag{3.68}$$

Then eq. (3.67) implies the massless KG equation. However, eq. (3.67) is a first-order differential equation, and gives further information. Consider for instance a plane wave solution of positive energy,

$$\psi_L(x) = u_L e^{-ipx} \tag{3.69}$$

where u_L is a constant spinor, and all the x-dependence is in the plane wave $\exp\{-ipx\} = \exp\{-iEt + i\mathbf{p}\cdot\mathbf{x}\}$. Then eq. (3.67) gives

$$\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E} u_L = -u_L \tag{3.70}$$

and eq. (3.68) gives $E = |\mathbf{p}|$. Since for a spin 1/2 field the angular momentum is $\mathbf{J} = \boldsymbol{\sigma}/2$, eq. (3.70) can be rewritten as

$$(\hat{\mathbf{p}} \cdot \mathbf{J})\, u_L = -\frac{1}{2} u_L \,, \tag{3.71}$$

where $\hat{\mathbf{p}} = \mathbf{p}\,/|\mathbf{p}\,|$. This shows that *a left-handed massless Weyl spinor has helicity* $h = -1/2$. This result is consistent with our discussion of the representations of the Poincaré group in Chapter 2, where we found that massless particles are helicity eigenstates.[8]

The energy–momentum tensor is obtained from the general formula (3.35). Observe that on a classical solution $\bar{\sigma}^\mu \partial_\mu \psi_L = 0$, so the Lagrangian (3.66) vanishes. The energy–momentum tensor is therefore

$$\theta^{\mu\nu} = i\psi_L^\dagger \bar{\sigma}^\mu \partial^\nu \psi_L \,, \tag{3.72}$$

and in particular

$$\theta^{00} = i\psi_L^\dagger \partial_0 \psi_L \,. \tag{3.73}$$

The Lagrangian (3.66) is invariant under a global $U(1)$ internal transformation,

$$\psi_L \to e^{i\theta} \psi_L \,, \tag{3.74}$$

[8] Let us anticipate that, when we quantize the theory, this result will translate into the existence of massless quanta of the field ψ_L with helicity $h = -1/2$, while the negative energy solutions will correspond to antiparticles with $h = +1/2$; see Section 4.2.2.