Nothing but Relativity

Palash B. Pal

Saha Institute of Nuclear Physics, 1/AF Bidhan-Nagar, Calcutta 700064, INDIA

Abstract

We deduce the most general space-time transformation laws consistent with the principle of relativity. Thus, our result contains the results of both Galilean and Einsteinian relativity. The velocity addition law comes as a bi-product of this analysis. We also argue why Galilean and Einsteinian versions are the only possible embodiments of the principle of relativity.

Historically, Einstein's special theory of relativity was motivated by considerations of properties of light. Even today, textbooks and other expositions of the special theory rely heavily on gedanken experiments involving light. The Lorentz transformation equations, the formula for relativistic addition of velocities, and other important formulas of the special theory are derived using light signals.

There are two assumptions, or axioms, underlying the special theory of relativity. One is the principle of relativity, which asserts that physical laws appear the same to any inertial observer. The other, which marks the difference of Einstein's theory with the earlier Galilean theory of relativity, is the assertion of the constancy of the speed of light in the vacuum.

An interesting question to ask, therfore, is the following. Suppose one takes the principle of relativity, but does not take the second axiom of Einstein. One would then obtain the most general formulas equations which are consistent with the principle of relativity. Such formulas would contain both Galilean and Einsteinian results. This question has been asked before in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], and the authors have derived the relativistic velocity addition law in some cases, the space-time transformation equations in some other. Here, we present an approach to the same problem which is somewhat different, and at the end, both the space-time transformations and the velocity addition law come out from the same exercise.

Let us consider two inertial frames S and S', where the second one moves with a speed v, along the x-axis, with respect to the first one. The co-ordinates and time in the S-frame will be denoted by x and t, and in the frame S', they will be denoted with a prime. The space-time transformation equations have the form

$$x' = X(x,t,v)\,, \tag{1}$$
$$t' = T(x,t,v)\,, \tag{2}$$

and our task is to determine these functions. A few properties of these functions can readily be observed. First, the principle of relativity tells us that if we invert these equations, we must obtain the same functional forms:

$$x = X(x',t',-v)\,, \tag{3}$$
$$t = T(x',t',-v)\,. \tag{4}$$

Notice that here the third argument of the functions is $-v$, since that is the velocity of the frame S with respect to S'. Using Eqs. (1) and (2) now, we can rewrite Eqs. (3) and (4) as:

$$x = X\left(X(x,t,v),\, T(x,t,v),\, -v\right)\,, \tag{5}$$
$$t = T\left(X(x,t,v),\, T(x,t,v),\, -v\right)\,, \tag{6}$$

which are implicit constraints on the forms of the functions. Moreover, isotropy of space demands that we could take the x-axis in the reverse direction as well. In this case, both x and v change sign, and so does x'. In other words,

$$X(-x,t,-v) = -X(x,t,v)\,, \tag{7}$$
$$T(-x,t,-v) = T(x,t,v)\,. \tag{8}$$

We now invoke the homogeneity of space and time. Suppose there is a rod placed along the x-axis such that its ends are at points x_1 and x_2 in the frame S, with $x_2 > x_1$. In the frame S', the ends will be at the points $X(x_1,t,v)$ and $X(x_2,t,v)$, so that the length would be

$$l' = X(x_2,t,v) - X(x_1,t,v)\,. \tag{9}$$

Suppose we now displace the rod such that its end which used to be at x_1 is now at the point $x_1 + h$. Its length in the frame S should not be affected by its position on the x-axis by virtue of the principle of homogeneity of space, so that its other end should now be at the point $x_2 + h$. In the frame S', its ends will be at the points $X(x_2 + h,t,v)$ and $X(x_1 + h,t,v)$. However, homogeneity of space implies that the length of the rod should not be affected in the frame S' as well, so that

$$l' = X(x_2 + h,t,v) - X(x_1 + h,t,v)\,. \tag{10}$$

Using Eqs. (9) and (10), we obtain

$$X(x_2 + h,t,v) - X(x_2,t,v) = X(x_1 + h,t,v) - X(x_1,t,v)\,. \tag{11}$$

Dividing both sides by h and taking the limit $h \to 0$, we obtain

$$\left.\frac{\partial X}{\partial x}\right|_{x_2} = \left.\frac{\partial X}{\partial x}\right|_{x_1} . \tag{12}$$

Since the points x_2 and x_1 are completely arbitrary, this implies that the partial derivative $\partial X/\partial x$ is constant, independent of the point x. Thus, the function $X(x,t,v)$ must be a linear function of x. One can similarly argue, invoking the homogeneity of time as well, that both $X(x,t,v)$ and $T(x,t,v)$ are linear in the arguments x and t. In that case, making the trivial choice that the origins of the two frames coincide, i.e., $x = t = 0$ implies $x' = t' = 0$, we can write

$$X(x,t,v) = A_v x + B_v t \, , \tag{13}$$

$$T(x,t,v) = C_v x + D_v t \, , \tag{14}$$

where the subscript v on the co-efficients A, B, C and D remind us that they are functions of the relative velocity v only. Eqs. (7) and (8) then imply that

$$A_{-v} = A_v \, , \quad B_{-v} = -B_v \, , \quad C_{-v} = -C_v \, , \quad D_{-v} = D_v \, . \tag{15}$$

In other words, A and D are even functions, while B and C are odd functions of v. Using these properties, we can now use Eqs. (5) and (6) to obtain the following conditions:

$$A_v^2 - B_v C_v = 1 \, , \tag{16}$$

$$B_v(A_v - D_v) = 0 \, , \tag{17}$$

$$C_v(A_v - D_v) = 0 \, , \tag{18}$$

$$D_v^2 - B_v C_v = 1 \, . \tag{19}$$

Unfortunately, these four equations do not solve the four functions A, B, C and D. The reason is simple. Eqs. (17) and (18) indicate two possibilities. Either $B_v = C_v = 0$, in which case the other two equation say that $A_v = D_v = 1$, which is just the trivial solution of identity transformation. While this is mathematically a valid possiblity, physically it is not acceptable for arbitrary values of v. Thus we look at the other case, which gives

$$D_v = A_v \, , \tag{20}$$

$$C_v = \frac{A_v^2 - 1}{B_v} \, . \tag{21}$$

Thus, two of the functions of v introduced in Eqs. (13) and (14) are independent.

In fact, we can reduce the number of independent functions further if we notice that by our definition, the origin of the frame S' is moving at a speed v with respect to the origin of S, i.e., at time t, it must be at the point $x = vt$. In other words, $x' = 0$ when $x = vt$. This implies

$$B_v = -v A_v \, , \tag{22}$$

so that now we can write down the transformation equation in terms of just one unknown function A_v, this time in a matrix notation:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} A_v & -vA_v \\ -\frac{A_v^2-1}{vA_v} & A_v \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} . \tag{23}$$

So far, the functional form of A_v is unknown, except for the fact that it is an even function of v, and that it must equal unity when $v = 0$. However, we can go further if we now consider a third frame S'' which is moving with a speed u with respect to S'. Then

$$\begin{aligned} \begin{pmatrix} x'' \\ t'' \end{pmatrix} &= \begin{pmatrix} A_u & -uA_u \\ -\frac{A_u^2-1}{uA_u} & A_u \end{pmatrix} \begin{pmatrix} A_v & -vA_v \\ -\frac{A_v^2-1}{vA_v} & A_v \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \\ &= \begin{pmatrix} A_uA_v + (A_v^2-1)\frac{uA_u}{vA_v} & -(u+v)A_uA_v \\ -(A_u^2-1)\frac{A_v}{uA_u} - (A_v^2-1)\frac{A_u}{vA_v} & A_uA_v + (A_u^2-1)\frac{vA_v}{uA_u} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \end{aligned} \tag{24}$$

However, Eq. (20) tells us that the two diagonal elements of this matrix should be equal, which implies

$$\frac{A_v^2-1}{v^2A_v^2} = \frac{A_u^2-1}{u^2A_u^2} . \tag{25}$$

But the left side of this equation depends only on v, while the right side depends only on u. They can be equal only if they are constants. Denoting this constant by K, we obtain

$$A_v = \frac{1}{\sqrt{1-Kv^2}} . \tag{26}$$

Using this form in Eq. (23), we thus obtain that the most general transformation equations consistent with the principle of relativity are of the form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1-Kv^2}} \begin{pmatrix} 1 & -v \\ -Kv & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} . \tag{27}$$

Another thing to notice is that the velocity addition law can be directly deduced from our analysis. For this, let us call the speed of the frame S'' with respect to S by w. Then, in Eq. (24), the diagonal terms of the matrix must be A_w:

$$\begin{aligned} A_w &= A_uA_v + (A_v^2-1)\frac{uA_u}{vA_v} \\ &= A_uA_v(1+Kuv) , \end{aligned} \tag{28}$$

using in the last step the definition of K which follows from Eq. (25). Given the form of the function A from Eq. (26), it is now easy to deduce that

$$w = \frac{u+v}{1+Kuv} , \tag{29}$$

which is the velocity addition law.

Specific theories of relativity, of course, have to make extra assumptions in order to determine the value of K. In the case of Galilean relativity, this extra assumption shows up in the form of the universality of time, which means $t' = t$ for any v. Obviously, this requires $K = 0$. The extra assumption for Einstein's theory of relativity is the constancy of the speed of light in vacuum. From Eq. (29), it is easy to see that $K^{-1/2}$ is an invariant speed, independent of the frame of reference. Thus, $K = 1/c^2 > 0$ in this case. It is obvious that in both these cases, we obtain the appropriate transformation laws from Eq. (27) and the velocity addition law from Eq. (29).

From this line of reasoning, it seems that there should be another logical possibility with $K < 0$. Actually, this option is not self-consistent. To see this, we first look at Eq. (26), and note that only the positive square root can be taken in the expression on the right hand side, because we want A_v to reduce to unity when v vanishes. Thus, $A_v \geq 0$ for any v. However, if K is negative, i.e., $K = -1/C^2$ for some finite value of C, we can obtain $A_w < 0$ from Eq. (28) if we choose large enough values of u and v which satisfy $uv > C^2$.

One point has to be made here. For the case of Einsteinian relativity as well, one can reach a contradiction, viz., that A_v becomes imaginary if $v > c$. But such large speeds are unreachable in Einsteinian relativity due to the structure of the addition law of Eq. (29), which shows that one cannot obtain $w > c$ if both u and v are less than c. For $K = -1/C^2$, this is not the case. One can add two speeds, both less than C, and the result of addition can be larger than C. For example, if the speed of S' is $C/2$ with respect to S, and if S'' moves with a speed $C/2$ with respect to S', the speed of S'' from the S-frame is $4C/3$. Thus, speeds larger than C cannot be excluded from this theory, but such speeds raise the possibility of having $A_w < 0$ as outlined above. Hence the inconsistency.

Thus, in effect, we have deduced the most general space-time transformation law as well as the velocity addition law consistent with the principle of relativity, and have shown that Galilean and Einsteinian laws are the only possible ones. Our method most closely resembles that of Singh [10], but there are important differences. In his derivation, Singh used some properties of the velocity addition law deduced by Mermin [7]. We have not used them. On the other hand, we have made direct use of the isotropy of space to deduce the symmetry properties of the functions A, B, C and D which have been summarized in Eq. (15) and used them to obtain Eqs. (16-19). But the most important difference, to our mind, is that while previous derivations used distinct lines of reasoning for the space-time transformation laws and the velocity addition formula, our argument gives both at the same stroke.

Acknowledgements: I thank P. Bhattacharjee, B. P. Das, J. Samuel, D. Sen and S. Sinha for patiently listening to my arguments, commenting on them, and suggesting some references which I could have missed otherwise.

References

[1] R. Weinstock: *New approach to special relativity*, Am. J. Phys. 33 (1965) 540—645.

[2] V. Mitavalský: *Special relativity without the postulate of constancy of light*, Am. J. Phys. 34 (1966) 825.

[3] A. R. Lee, T. M. Kalotas: *Lorentz transformations from the first postulate*, Am. J. Phys. 43 (1975) 434—437.

[4] J.-M. Lévy-Leblond: *One more derivation of the Lorentz transformation*, Am. J. Phys. 44 (1976) 271—277.

[5] W. Rindler: *Essential Relativity* (Springer-Verlag, 2nd edition, 1977). See §2.17.

[6] A. M. Srivastava: *Invariant speed in special relativity*, Am. J. Phys. 49 (1981) 504—505.

[7] N. D. Mermin: *Relativity without light*, Am. J. Phys. 52 (1984) 119—124.

[8] H. M. Schwartz: *Deduction of the general Lorentz transformations from a set of necessary assumptions*, Am. J. Phys. 52 (1984) 346—350.

[9] H. M. Schwartz: *A simple new approach to the deduction of the Lorentz transformations*, Am. J. Phys. 53 (1985) 1007—1008.

[10] S. Singh: *Lorentz transformations in Mermin's relativity without light*, Am. J. Phys. 54 (1986) 183-184.

[11] A. Sen: *How Galileo could have derived the special theory of relativity*, Am. J. Phys. 62 (1994) 157-162.